AMERICAN JOURNAL OF MATHEMATICS

FOUNDED BY THE JOHNS HOPKINS UNIVERSITY

PUBLISHED UNDER THE JOINT AUSPICES OF
THE JOHNS HOPKINS UNIVERSITY
AND
THE AMERICAN MATHEMATICAL SOCIETY

Volume LXVI, Number 3
JULY, 1944

THE JOHNS HOPKINS PRESS
BALTIMORE 18, MARYLAND
U. S. A.

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The AMERICAN JOURNAL OF MATHEMATICS will appear four times yearly.

The subscription price of the JOURNAL for the current volume is \$7.50 (foreign postage 50 cents); single numbers \$2.00.

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Subscriptions to the JOURNAL and all business communications should be sent to THE JOHNS HOPKINS PRESS, BALTIMORE 18, MARYLAND, U. S. A.

Entered as second-class matter at the Baltimore, Maryland, Postoffice, acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, Authorized on July 3, 1918.

PRINTED IN THE UNITED STATES OF AMERICA
BY J. H. FURST COMPANY, BALTIMORE, MARYLAND

CROSSED ISOMORPHISMS.*

By Reinhold Baer.

Introduction. Isomorphisms are $1:1$ and multiplicative correspondences between the elements of groups. If one weakens the first of these two properties so as to consider single valued multiplicative functions, one obtains homomorphisms. An example of a transformation where the first property has been preserved, but not the multiplicativity, is the left-translation: $x \rightarrow ax$ for a a fixed and x the variable group element. These satisfy the functional equation: $(xy)' = x'a^{-1}y'$. Another type of functional equation, first investigated by I. Schur and much considered recently,[1] is the following: $x'y' = (x^{e(y)}y)'$ where $e(y)$ is, for every y, an endomorphism. We shall term the $1:1$ solutions of such a functional equation crossed isomorphisms or, if we want to emphasize the function $e(x)$ involved, $e(x)$-isomorphisms.[2] Special cases of crossed isomorphisms are the ordinary isomorphisms: $e(x) = 1$, and the anti-isomorphisms: $x^{e(y)} = yxy^{-1}$.

The problems we are concerned with may be grouped roughly by putting emphasis upon the transformation or by laying more stress on the group transformed. Fundamental questions of the first kind are to characterize the ordinary isomorphisms among the crossed isomorphisms and the latter among the $1:1$ correspondences. Investigations of the second type are concerned with finding group theoretical properties invariant under crossed isomorphisms and with the enumeration of all the groups obtainable from a given group by means of crossed isomorphisms. It is natural that crossed isomorphisms may serve as a tool for classifying groups. But these problems are so interrelated that it seems impossible to keep them apart. For answers to problems of the first kind will depend on the structure of the underlying group; and various classifications of groups may be obtained by restricting the crossed isomorphisms admitted.

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* Received August 6, 1943. Presented to the American Mathematical Society, April 29, 1944.

[1] Notably by A. H. Clifford, S. MacLane, O. F. G. Schilling and the author. For further references, see Baer (2), footnote [3]. The present investigation is, both in its problems and in its results, fairly independent of these papers.

[2] It has been customary to term the inverse of an $e(x)$-isomorphism a crossed isomorphism. But for our purposes the $e(x)$-isomorphisms, but not their inverses, are the transformations of primary interest.

The present investigation offers contributions towards a solution of all these problems. Since only few of the interesting group theoretical properties are left invariant by the most general crossed isomorphisms, we restrict ourselves mainly to a consideration of integral crossed isomorphisms which map cosets upon cosets and of the coset preserving crossed isomorphisms which have the property that the image of a set S is a coset if, and only if, S itself is a coset. The latter are characterized among the former by their conformality. If γ is an integral crossed isomorphism, then xy^{-1} and $(x^\gamma(y^\gamma)^{-1})^{\gamma^{-1}}$ generate the same cyclic subgroup; and coset preserving crossed isomorphisms map the cyclic subgroup, generated by xy^{-1}, upon the cyclic subgroup, generated by $x^\gamma(y^\gamma)^{-1}$. If the groups under consideration meet certain not too stringent requirements, then these properties may serve as characterizations among the $1:1$ correspondences.

Clearly the coset preserving crossed isomorphisms induce projectivities of a very strict type in the partially ordered set of all the subgroups; and so there arises naturally the further problem of obtaining given projectivities from crossed isomorphisms.

We conclude this introduction by indicating how the various problems mentioned so far are distributed over the present investigation. In **3** we characterize the coset preserving crossed isomorphisms among the integral crossed isomorphisms; and in **4** both integral and coset preserving crossed isomorphisms are characterized among the $1:1$ correspondences. Criteria for a crossed isomorphism to be an ordinary isomorphism may be found everywhere (Theorems 3. 3, 4. 1, 7. 3, 7. 4 and Corollaries 4. 2 and 6. 4). In **5**, **6**, **7** we are concerned with the construction of crossed isomorphisms which induce a given projectivity; and in **2**, **7**, **8**, **9** we determine the groups obtainable from abelian groups by means of integral crossed isomorphisms. We mention, finally, Theorem 6. 3 which gives a criterion for a group to be abelian; that it is not strange to find such a theorem here, may be seen from the obvious remark that a group is abelian if, and only if, the inversion: $x \rightarrow x^{-1}$ is an ordinary automorphism.

Throughout we shall denote by $\{x, y, \cdots\}$ the subgroup, generated by the elements $x, y, \cdots$, and by $S \cap T$ the cross cut of the sets S and T.

1. Fundamental concepts. The $1:1$ correspondence γ mapping the elements in the group A upon the elements in the group B is termed *a crossed isomorphism of A upon B, if $A^\gamma = B$, and if there exists to every element w in A an endomorphism*[3] *$e(w)$ of A such that $u^\gamma v^\gamma = (u^{e(v)}v)^\gamma$ for u and v*

[3] An endomorphism is a homomorphism of a group into itself.

in A. It is readily seen that the function $e(w)$, entering into the definition of the crossed isomorphism γ is uniquely determined by γ. It will often be convenient to stress this function $e(x)$ by terming γ an *$e(x)$-isomorphism of A upon B.* If γ and κ are both $e(x)$-isomorphisms of the group A upon the groups G and K respectively, then $(u^\gamma v^\gamma)^{\gamma^{-1}\kappa} = u^k v^k$, showing that $\gamma^{-1}\kappa$ is an ordinary isomorphism of the group G upon the group K. The structure of the image of the group A under a crossed isomorphism is therefore uniquely determined by the function $e(w)$. The $e(x)$-isomorphism γ of the group A is, in particular, an ordinary isomorphism if, and only if, $e(x) = 1$ for every x in A. It should be noted, however, that it will in general be impossible[4] to prove that products or inverses of crossed isomorphisms are crossed isomorphisms.

THEOREM 1.1. *If $e(x)$ is, for every element x in the group A, an endomorphism of the group A, then the following conditions are necessary and sufficient for the existence of an $e(x)$-isomorphism of A upon some group B:*

(a) $e(u)e(v) = e(u^{e(v)}v)$ *for u and v in A.*

(b) *Every $e(x)$ is a proper automorphism of A.*

(c) $e(1) = 1$.

Proof. Suppose, first, that there exists an $e(x)$-isomorphism γ of the group A upon some group B. If t, u, v are elements in A, then we deduce from the associative law in B and from the multiplicativity of the endomorphism $e(x)$ that

$$(t^{e(u)e(v)}u^{e(v)}v)^\gamma = ((t^{e(u)}u)^{e(v)}v)^\gamma = (t^{e(u)}u)^\gamma v^\gamma = t^\gamma u^\gamma v^\gamma$$
$$= t^\gamma (u^{e(v)}v)^\gamma = (t^{e(u^{e(v)}v)}u^{e(v)}v)^\gamma.$$

Since γ is a $1:1$ correspondence we may infer from the cancellation law in A that $t^{e(u)e(v)} = t^{e(u^{e(v)}v)}$ for every t in A, proving the necessity of (a). If u and v are elements in A, then there exists one and only one element x in A such that $x^\gamma u^\gamma = v^\gamma$, since $B = A^\gamma$. Consequently there exists one and only one element x in A such that $x^{e(u)}u = v$, proving the necessity of (b). From $1^\gamma x^\gamma = (1^{e(x)}x)^\gamma = x^\gamma$ we deduce $1^\gamma = 1$; and from $x^\gamma = x^\gamma 1^\gamma = (x^{e(1)}1)^\gamma = (x^{e(1)}1)^\gamma$ we infer that $x = x^{e(1)}$ for every x, showing that $e(1) = 1$, and completing the proof of the necessity of the three conditions.

If, conversely, the conditions (a) to (c) are satisfied by $e(x)$, then we define a new multiplication: $x \circ y$ of the elements in A by the formula:

$$x \circ y = x^{e(y)}y \text{ for } x \text{ and } y \text{ in } A.$$

[4] See the Theorems 1.4′ and 1.4″ below.

It is obvious that this new multiplication is unique. If t, u, v are elements in A, then

$$\begin{aligned}(t \circ u) \circ v &= (t^{e(u)}u) \circ v = (t^{e(u)}u)^{e(v)}v = t^{e(u)e(v)}u^{e(v)}v \\ &= t^{e(u^{e(v)}v)}(u^{e(v)}v) = t \circ (u^{e(v)}v) = t \circ (u \circ v),\end{aligned}$$

since $e(v)$ is a homomorphism and since (a) may be applied, proving the associativity of the new multiplication. The element x in A is a solution of the equation $x \circ u = v$ if, and only if, $x^{e(u)}u = v$ or $x^{e(u)} = vu^{-1}$; and this last equation has one and only one solution x, since $e(u)$ is by (b) a proper automorphism of A. From condition (c) we derive, finally, that $1 \circ u = (1^{e(u)}u) = u$ and $u \circ 1 = u^{e(1)}1 = u$, i. e. 1 is the unit element under the new multiplication too. But it is well known that from the facts derived so far the group property of the new multiplication in A may be deduced.

Thus A is a group both under the multiplication: uv and under the multiplication $u \circ v = u^{e(v)}v$. But it is immediately obvious that the identity transformation is an $e(x)$-isomorphism of the group with the multiplication uv upon the group with the multiplication $u \circ v$; and this completes the proof of the theorem.

Corollary 1. 2. *If $e(x)$ is, for every x in the group A, an endomorphism of A, and if γ is an $e(x)$-isomorphism of A upon the group B, then γ has the following properties:*

(i) $e(v)^{-1} = e(v^{-e(v)^{-1}})$ *and* $(v^\gamma)^{-1} = (v^{-e(v)^{-1}})^\gamma$ *for v in A.*

(ii) *The set E of all the elements x in A satisfying $e(x) = 1$ is a subgroup of A; and $e(u) = e(v)$ if, and only if, $u^{-1}v$ is in E.*

(iii) *γ induces an ordinary isomorphism of E upon a normal subgroup E^γ of B; and a homomorphism of B with kernel[5] E^γ is effected by mapping the element y in B upon the endomorphism $e(y^{\gamma^{-1}})$ of A.*

(iv) $(v^{-1}u^{e(v)}v)^\gamma = (v^\gamma)^{-1}u^\gamma v^\gamma$ *for u in E and v in A; and if v is an element in A, then mapping the element u in E upon the element $v^{-1}u^{e(v)}v$ constitutes an automorphism of E.*

Remark. It is readily seen that a function $e(x)$, satisfying the first part of (i), satisfies the conditions (a) to (c) of Theorem 1. 1, provided it satisfies (a).

Proof. It is a consequence of Theorem 1. 1 that $e(v)^{-1}$ is, for every v in A, a well determined automorphism of A, and that

[5] The kernel of a homomorphism consists of the elements mapped upon 1; it is always a normal subgroup.

$$1 = e(1) = e(v^{-1}v) = e(v^{-e(v)^{-1}e(v)}v) = e(v^{-e(v)^{-1}})e(v),$$

proving the first part of (i). Consequently we derive from the definition of crossed isomorphisms that

$$v^{\gamma}(v^{-e(v)^{-1}})^{\gamma} = (v^{e(v^{-e(v)^{-1}})}v^{-e(v)^{-1}})^{\gamma} = (v^{e(v^{e(v)^{-1}})-e(v)^{-1}})^{\gamma} = 1,$$

completing the proof of (i), since B is a group.

If v and w are elements in A, then we deduce from (i) and Theorem 1.1, (a) that $e(v) = e(w)$ is a necessary and sufficient condition for

$$1 = e(v)^{-1}e(w) = e(v^{-e(v)^{-1}})e(w) = e(v^{-e(v)^{-1}e(w)}w) = e(v^{-1}w).$$

Hence $e(v) = e(w)$ if, and only if, $v^{-1}w$ is in E; and (ii) is an immediate consequence of this fact.

If x and y are elements in B, then we infer from the definition of crossed isomorphisms and from Theorem 1. 1 that

$$e(x^{\gamma^{-1}})e(y^{\gamma^{-1}}) = e((x^{\gamma^{-1}})^{e(y^{\gamma^{-1}})}y^{\gamma^{-1}}) = e((xy)^{\gamma^{-1}}).$$

If u is an element in E, v in A, then we have $(vu)^{\gamma} = (v^{e(u)}u)^{\gamma} = v^{\gamma}u^{\gamma}$; and (iii) is an immediate consequence of these two equations.

If u is an element in E and v an element in A, then we deduce from (i) and Theorem 1. 1 that

$$\begin{aligned}(v^{-1}u^{e(v)}v)^{\gamma} &= ((v^{-e(v)^{-1}}u)^{e(v)}v)^{\gamma} = (v^{-e(v)^{-1}}u)^{\gamma}v^{\gamma} \\ &= (v^{-e(v)^{-1}})^{\gamma}u^{\gamma}v^{\gamma} = (v^{\gamma})^{-1}u^{\gamma}v^{\gamma}.\end{aligned}$$

It is a consequence of (iii) that E^{γ} is a normal subgroup of B; and hence $v^{-1}u^{e(v)}v$ belongs to E. This completes the proof of (iv).

THEOREM 1. 3. *Suppose that $e(x)$ is, for every x in the group A, an endomorphism of A, that γ is an $e(x)$-isomorphism of A upon the group B, and that S is a subgroup of A. Then*

(1) *$S^{e(s)} = S$ for every s in S if, and only if, S^{γ} is a subgroup of B; and*

(2) *$S^{e(x)} = S$ for every x in A if, and only if, $u^{\gamma}(v^{\gamma})^{-1}$ belongs to S^{γ} whenever uv^{-1} is in S.*

Proof. If u and v are elements in A, then we deduce from Theorem 1. 1 the existence of one and only one element w in A such that $w^{e(v)} = uv^{-1}$; and hence it follows from the definition of $e(x)$-isomorphisms and Theorem 1. 1 that

$$w^{\gamma}v^{\gamma} = (w^{e(v)}v)^{\gamma} = (uv^{-1}v)^{\gamma} = u^{\gamma}.$$

Consequently $u^\gamma(v^\gamma)^{-1} = w^\gamma = ((uv^{-1})^{e(v)^{-1}})^\gamma$; and $u^\gamma(v^\gamma)^{-1}$ belongs to S^γ if and only if, $(uv^{-1})^{e(v)^{-1}}$ is in S.

From this last equivalence one deduces now that S^γ is a subgroup of B if and only if, $S^{e(s)^{-1}} \leq S$ for every s in S; and that $u^\gamma(v^\gamma)^{-1}$ belongs to S^γ whenever uv^{-1} is in S if, and only if, $S^{e(x)^{-1}} \leq S$ for every x in A. It is a consequence of Corollary 1. 2, (iii) that the $e(t)$ for t in the subgroup T of A form a group of endomorphisms of A, provided that T^γ is a subgroup of B; and now our contentions are readily verified, using Corollary 1. 2, (i).

An endomorphism e of the group A will be termed *an integral endomorphism of A, if $S^e \leq S$ for every subgroup S of A*; and likewise we term an automorphism f of A *an integral automorphism of A, if $S^f = S$ for every subgroup S of A*. It should be noted that an integral endomorphism which happens to be at the same time an automorphism need not be an integral automorphism (as an example map every rational number r upon $2r$). If f is an integral automorphism of A, Z a cyclic subgroup of A, then there exists an integer $i = i(Z)$ such that $z^f = z^i$ for every z in Z, and such that z^i generates Z whenever z generates Z; and this property is characteristic for the integrality of an automorphism. Consequently the integral automorphisms of A form an abelian group.

If $e(x)$ is, for every x in the group A, an integral automorphism of A and if γ is an $e(x)$-isomorphism of A upon the group B, then we term γ an *integral crossed isomorphism of A upon B*. It is readily deduced from Theorem 1. 3 that the crossed isomorphism γ of the group A upon the group B is an integral crossed isomorphism if, and only if, it meets the following requirement:

(1. 3. 3) *If S is a subgroup of A and if u and v are elements in A such that uv^{-1} belongs to S, then S^γ is a subgroup of B and $u^\gamma(v^\gamma)^{-1}$ belongs to S^γ.*

That integral crossed isomorphisms are *not* the only crossed isomorphisms which map subgroups upon subgroups, may be seen from the example of the inversion $x' = x^{-1}$ which is a crossed isomorphism, mapping subgroups upon subgroups, since $x'y' = (x^{e(y)}y)'$ where $e(y)$ indicates the inner automorphism induced by y^{-1}, which in general is not an integral automorphism.

It has been remarked before that the products and inverses of crossed isomorphisms cannot be expected to be crossed isomorphisms too. The following theorems may serve as a substantiation for this contention.

THEOREM 1. 4′. *If there exists an $e(x)$-isomorphism of the group A upon some group B, then the following properties imply each other:*

(I) *The inverse of an $e(x)$-isomorphism is a crossed isomorphism.*

(II) $e(uv) = e(v)e(u)$ *for u and v in A.*

Remark. Condition (II) states that $e(x)$ is an anti-homomorphism of A into the group of automorphisms of A. Combining (II) and Theorem 1.1, (a) we obtain:

$$\begin{aligned} e(uv)e(u^{e(v)-1}) &= e(u^{e(v)}v) = e(u)e(v) = e(vu) \\ &= e(vuv^{-1}u^{-1}uv) = e(uv)e(vuv^{-1}u^{-1}) \end{aligned}$$

or $e(vuv^{-1}u^{-1}) = e(u^{e(v)-1})$ by Theorem 1.1, (b). If, in particular, A is abelian or if the automorphisms $e(x)$ commute with each other, then $e(u^{e(v)-1}) = 1$ for u, v in A.

Proof. There exists an $e(x)$-isomorphism γ of A upon some group B. Denote by η the inverse γ^{-1} of γ. If u and v are elements in B, then u^η and v^η are elements in A; and we deduce from the definition of an $e(x)$-isomorphism and from Theorem 1.1, (b) that

$$u^\eta v^\eta = ((u^\eta)^{e(v^\eta)^{-1}e(v^\eta)}v^\eta)^{\gamma\eta} = (u^{\eta e(v^\eta)^{-1}\gamma}v)^\eta = (u^{f(v)}v)^\eta$$

where $f(v) = \eta e(v^\eta)^{-1}\gamma$ is a well determined 1:1 transformation of B upon itself.

Clearly η is a crossed isomorphism if, and only if, $f(v)$ is an endomorphism of B, for every v. If x and y are elements in B, then

$$(xy)^{f(v)} = x^{\eta e(y^\eta)e(v^\eta)^{-1}e(y^{\eta e(v^\eta)-1})^{-1}\gamma}y^{f(v)}.$$

Consequently $f(v)$ is an endomorphism if, and only if,

$$e(v^\eta)^{-1} = e(y^\eta)e(v^\eta)^{-1}e(y^{\eta e(v^\eta)^{-1}})^{-1} \text{ for every } y \text{ in } B.$$

Hence (I) is satisfied if, and only if, we have for every r, s in A:

$$e(r)e(s) = e(s^{e(r)^{-1}})e(r) = e(sr)$$

by Theorem 1.1. (a), completing the proof.

THEOREM 1.4″. *If $a(x)$ is, for every element x in the group A, an endomorphism of A, if $b(y)$ is, for every element y in the group B, an endomorphism of B, if ϕ is an $a(x)$-isomorphism of A upon B, and if γ is a $b(x)$-isomorphism of B upon some group G, then*

$$\phi b(y)\phi^{-1}a(z^{b(y)\phi^{-1}}) = a(z^{\phi^{-1}})\phi b(y)\phi^{-1} \textit{ for } y, z \textit{ in } B$$

is a necessary and sufficient condition for $\phi\gamma$ to be a crossed isomorphism of A upon G.

Remark 1. The above condition is trivially satisfied, if ϕ or γ is an ordinary isomorphism, since in the first case $a(x) = 1$ for every x, and since in the second case $b(y) = 1$ for every y.

Remark 2. *If $a(x)$ and $\phi b(y)\phi^{-1}$ permute for every x and y, then the condition of the theorem reduces to $a(z^{\phi^{-1}}) = a(z^{b(y)\phi^{-1}})$.*

Proof. If r and s are elements in A, then

$$r^{\phi\gamma}s^{\phi\gamma} = (r^{\phi b(s^\phi)}s^\phi)^\gamma = (r^{\phi b(s^\phi)\phi^{-1}\phi}s^\phi)^\gamma = (r^{\phi b(s^\phi)\phi^{-1}a(s)}s)^{\phi\gamma}.$$

This shows that $\phi\gamma$ is a crossed isomorphism of A upon G if, and only if, $\phi b(s^\phi)\phi^{-1}a(s)$ is an endomorphism of A for every s in A.

If u and v are elements in A, then

$$\begin{aligned}(u^{a(v)}v)^{\phi b(s^\phi)\phi^{-1}a(s)} &= (u^{\phi b(s^\phi)}v^{\phi b(s^\phi)})^{\phi^{-1}a(s)}\\ &= (u^{\phi b(s^\phi)\phi^{-1}a(v^{\phi b(s\phi)\phi^{-1}})}v^{\phi b(s^\phi)\phi^{-1}})^{a(s)}\\ &= u^{\phi b(s^\phi)\phi^{-1}a(v^{\phi b(s\phi)\phi^{-1}})a(s)}v^{\phi b(s^\phi)\phi^{-1}a(s)}\end{aligned}$$

and $\phi b(s^\phi)\phi^{-1}a(s)$ is therefore an endomorphism of A if, and only if,

$$a(v)\phi b(s^\phi)\phi^{-1}a(s) = \phi b(s^\phi)\phi^{-1}a(v^{\phi b(s^\phi)\phi^{-1}})a(s) \quad \text{for every } v \text{ in } A.$$

The validity of the theorem is now readily deduced from Theorem 1. 1, if one puts $s^\phi = y$ and $v^\phi = z$.

In part of this investigation, especially in sections **7** to **9**, we shall restrict ourselves to the consideration of crossed isomorphisms of abelian groups A. This hypothesis will not simplify the preceding discussion nor will it be possible to improve our results by means of the presupposed commutativity of A, with two exceptions which we proceed to discuss.

THEOREM 1. 5. *The endomorphism ϕ of the abelian group A is integral if, and only if, there exists to every pair of elements x, y in A an integer n such that $x^\phi = x^n$ and $y^\phi = y^n$.*

Proof. The sufficiency of this condition is immediately evident. To prove the necessity of the condition we consider elements x, y in A. They generate a subgroup which is the direct product of two cyclic groups, generated by elements u and v (one or both of which may be equal to 1). From the integrality of ϕ we deduce the existence of integers i, j, k such that $u^\phi = u^i$, $v^\phi = v^j$ and $(uv)^\phi = (uv)^k = u^k v^k$. Since ϕ is an endomorphism of an

abelian group, we have: $u^i v^j = u^\phi v^\phi = (uv)^\phi = u^k v^k$. From the independence of u and v we infer now the equalities: $u^\phi = u^i = u^k$ and $v^\phi = v^j = v^k$, proving that $x^\phi = x^k$ and $y^\phi = y^k$, as was to be shown.

COROLLARY 1. 6. *If γ and η are isomorphisms of the abelian group A upon the same group B, and if ϕ is an integral endomorphism of A, then $\gamma^{-1}\phi\gamma$ and $\eta^{-1}\phi\eta$ are equal integral endomorphisms of B.*

This is an immediate consequence of Theorem 1. 5. On the basis of this corollary it is possible to identify the integral endomorphisms in isomorphic abelian groups.

The set I_A *of all the integral endomorphisms of the abelian group A* is easily seen to be a *ring*; and the set I^*_A *of the integral automorphisms of A* is a multiplicative abelian group in the ring I_A.

If x is an element in a group, then we denote by $n(x)$ the *order* of x; and we note that the order of x is 0, if x generates an infinite cyclic group.

If the abelian group A contains elements of order 0, then it is readily deduced from Theorem 1. 5 that I_A is essentially the same as the ring of the rational integers; and the group I^*_A consists just of 1 and -1. If the abelian group A does not contain elements of order 0, and if the orders of the elements in A are bounded, then there exists a positive integer $m = m(A)$, the l. c. m. of the orders of the elements in A, such that I_A is essentially the same as the ring of rational integers modulo m; and I^*_A consists of those classes of residues modulo m which are relatively prime to m. A similar characterization may be given, in case the orders of the elements in A are not bounded. Suffice it to say that I_A is essentially the ring of p-adic integers, if all the elements in A are of order a power of p, and if the orders of the elements in A are not bounded; and I^*_A is in this case just the group of p-adic integers prime to p. Let us repeat that in every case I^*_A is commutative.

If A is an abelian group, if $e(x)$ is, for every x in A, an integral automorphism of A, and if γ is an $e(x)$-isomorphism of A upon some group B, then every $e(x)$ is an integral automorphism of the subgroup E of the elements t such that $e(t) = 1$ and γ induces an isomorphism of E upon the normal subgroup E^γ of B (Corollary 1. 2). From previous remarks it follows that $\gamma^{-1}e(y^{\gamma^{-1}})\gamma$ is, for every y in B, a well determined integral automorphism of E^γ which we may denote by $e(y)$ without any danger of confusion. Using this notation the statement (iv) of Corollary 1. 2 reads as follows:

COROLLARY 1. 7. *If $e(x)$ is, for every element x in the abelian group A, an integral automorphism of A, and if γ is an $e(x)$-isomorphism of A upon*

some group B, then B/E^{γ} and E^{γ} are abelian groups, and $v^{-1}uv = u^{e(v)}$ for u in E^{γ} and v in B.

Remark on crossed isomorphisms and crossed characters.[6]

Suppose that A is an abelian group without elements of order 0, that the orders of the elements in A are bounded and that m is the maximum order of the elements in A. If γ is an integral crossed isomorphism of A upon the group B, then there exists for every x in A an integral automorphism $e(x)$ of A such that γ is an $e(x)$-isomorphism. The function $d(x) = e(x^{\gamma^{-1}})$ for x in B is, by Corollary 1.2, a homomorphism of B into the group $I^*{}_A$ of the integers modulo m, prime to m. The characters of A are nothing but homomorphisms of A into a cyclic group of order m. If $h(x)$ is a character of A, then let $k(x) = h(x^{\gamma^{-1}})$ for x in B. One verifies readily that

$$\begin{aligned} k(xy) &= h((xy)^{\gamma^{-1}}) = h(x^{\gamma^{-1}e(y^{\gamma^{-1}})}y^{\gamma^{-1}}) = h(x^{\gamma^{-1}d(y)}y^{\gamma^{-1}}) \\ &= h(x^{\gamma^{-1}})^{d(y)}h(y^{\gamma^{-1}}) = k(x)^{d(y)}k(y), \end{aligned}$$

proving that k is a crossed (or $d(x)$-) character of B in the cyclic group of order m. Thus every character of A leads to a crossed character of B and different characters of A lead to different crossed characters of B; and the character group of A leads us to a complete[7] group of crossed characters of B. But this group of crossed characters of B need not be the group of *all* the crossed characters of B. If γ happens to be a coset preserving crossed isomorphism[8] of A upon B, then it may be shown that all the crossed characters of B are obtained in this fashion; and that consequently every result concerning the character group of A leads immediately to a result on the crossed character group of B. We note that, conversely, the correspondence F_g discussed in Baer (2), V. 1 is shown there to be quite often the inverse of a crossed isomorphism.

2. Functions $e(x)$ whose values are 1 and -1. In this section we shall give a complete theory of $e(x)$-isomorphisms in the special case where $e(x) = \pm 1$ for every x. The results will be needed later on. If $e(x) = 1$ for every x, then every $e(x)$-isomorphism is an ordinary isomorphism; and hence we shall usually exclude this possibility. If $e(x) = -1$ for some x, then A is necessarily commutative.

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[6] For the theory of crossed characters, see Baer (2).

[7] In the terminology of Baer (2), chapter IV.

[8] Anticipating a definition of section 3 below.

THEOREM 2.1. *Suppose that A is an abelian group, that $e(x) = \pm 1$ for every x in A and that $e(x) \neq 1$ for some x in A. Then there exists an $e(x)$-isomorphism of A upon some group if, and only if, the elements t in A which satisfy $e(t) = 1$ form a subgroup T of index 2 in A.*

Proof. If, firstly, T is a subgroup of index 2 in A, and if u and v are elements in A, then $u^{e(v)}v$ belongs to T if, and only if, either both u and v are in T or both u and v are not in T. Hence $e(u)e(v) = e(u^{e(v)}v)$ and the existence of an $e(x)$-isomorphism is a consequence of Theorem 1.1. Assume, conversely, that there exists an $e(x)$-isomorphism of A. Then it follows from Theorem 1.1 that $e(u)e(v) = e(u^{e(v)}v)$ for u and v in A. If, in particular, neither u nor v is in T, then $e(u) = e(v) = -1$ and hence $e(u^{-1}v) = e(u^{e(v)}v) = e(u)e(v) = 1$, showing that $u^{-1}v$ belongs to the subgroup T of A. Since T is supposed to be a proper subset of A, it follows that T is a subgroup of index 2 in A, as was to be shown.

COROLLARY 2.2. *The abelian group A satisfies: $1 < A^2 < A$ if, and only if, there exists a function $e(x)$ with the following properties: $e(x) = \pm 1$ for every x in A; there exists an element y in A such that $e(y) \neq 1$; there exists an $e(x)$-isomorphism of A upon some group.*

Proof. This is an immediate consequence of Theorem 2.1, if one remembers that the existence of a subgroup of index 2 in A is equivalent to $A^2 < A$; and that $A^2 = 1$ would imply $e(x) = 1$ for every x in A.

THEOREM 2.3. *Suppose that the group B is not abelian. Then there exists an abelian group A and a function $e(x)$ such that $e(x) = \pm 1$ for every x in A, $e(x) \neq 1$ for some x in A and such that B is an $e(x)$-isomorphic map of A if, and only if, there exists a subgroup S of index 2 in B such that $t^2 = 1$ for t not in S.*

Proof. If the group B contains a subgroup S of index 2 such that $t^2 = 1$ for t not in S, then S is a normal subgroup of B. If s is in S, and if t is not in S, then ts is not in S; and hence we find that $t^{-1}st = tst = tsts\,s^{-1} = (ts)^2s^{-1} = s^{-1}$. This implies, in particular, that S is an abelian group; and we may note that $S^2 \neq 1$, since B is not abelian. There exists obviously an abelian group A which contains S as a subgroup of index 2; and we put $e(x) = 1$ for x in S; $e(x) = -1$ for x in A, not in S. It is a consequence of Theorem 2.1 that there exists an $e(x)$-isomorphism γ of A upon some group B'. Patently S^γ is a subgroup of index 2 in B', as may be deduced from Theorem 1.3, (2); and if t is in A, but not in S, then $(t^\gamma)^2 = (t^{e(t)}t)^\gamma$

$=1$, showing the existence of an isomorphism of B' upon B which maps S^γ upon S; and thus we have proven the sufficiency of our condition. The necessity of the condition is shown by exactly the same arguments we have just used; and this completes the proof.

We note for future reference the following fact which we just proved:

COROLLARY 2.4. *If A is an abelian group, if T is a subgroup of index 2 in A, if $e(x)=1$ for x in T, $e(x)=-1$ for x not in T, and if γ is an $e(x)$-isomorphism of A upon a group, then γ maps the elements not in T upon elements of order 2.*

By arguments similar to those used in the proof of Theorem 2.3 it is possible to prove the following statement:

COROLLARY 2.5. *If the abelian group A contains a subgroup S of index 2, if the group B contains a subgroup T of index 2 such that the elements not in T are of order 2, and if S and T are isomorphic groups, then there exists a crossed isomorphism of A upon B which induces an ordinary isomorphism of S upon T.*

This shows the existence of abelian groups which are mapped by integral crossed isomorphisms upon not isomorphic abelian groups, the existence of not isomorphic abelian groups which may be mapped upon the same group by integral crossed isomorphisms; and that integral crossed isomorphisms need not preserve the orders of elements.[9]

3. Conformality and preservation of cosets. A 1:1 correspondence mapping the elements in the group A upon the elements in the group B is said to be *conformal*, if it maps every element in A upon an element of equal order in B.

The 1:1 correspondence γ mapping the elements in the group A upon the elements in the group B is termed *coset preserving* (or: γ preserves cosets), if it meets the following requirements:

(i) $A^\gamma = B$.

(ii) S is a subgroup of A if, and only if, S^γ is a subgroup of B.

(iii) uv^{-1} is in the subgroup S of A if, and only if, $u^\gamma(v^\gamma)^{-1}$ is in the subgroup S^γ of B.

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[9] The author is indebted to Drs. Beaumont and Zuckerman for pointing out to him this last possibility.

It is obvious that coset preserving correspondences between groups are conformal, since correspondences γ with the property (ii) are known to map cyclic subgroups upon cyclic subgroups.[10]

THEOREM 3.1. *The $e(x)$-isomorphism γ of the group A upon the group B preserves cosets if, and only if,*

(a) *γ is conformal; and*

(b) *γ is an integral crossed isomorphism of A upon B.*

Proof. The necessity of condition (a) has been pointed out before; and the necessity of condition (b) is a consequence of Theorem 1.3, (see condition (1.3.3), characterizing integral crossed isomorphisms).

Assume now that the conditions (a) and (b) are satisfied by the $e(x)$-isomorphism γ of A upon B. Then $A^\gamma = B$; and it follows from Theorem 1.3 that

(ii′) S^γ is a subgroup of B, if S is a subgroup of A; and that

(iii′) $u^\gamma(v^\gamma)^{-1}$ is in S^γ, if uv^{-1} is in the subgroup S of A.

We prove next the following lemma.

LEMMA (3.1.1). *$\{z\}^\gamma = \{z^\gamma\}$ for every z in A.*

γ induces by (ii′) an $e(x)$-isomorphism of the cyclic group $Z = \{z\}$ upon the subgroup $Y = Z^\gamma$ of B, where we may identify $e(x)$, for x in Z, with the integral automorphism of Z induced by $e(x)$.

Case 1. z is of order 0.

Then $e(x) = \pm 1$ on Z for every x in Z. If $e(x)$ were different from 1 for some x in Z, then we inferred from Theorem 2.1 that the elements t in Z with $e(t) = 1$ formed a subgroup T of index 2 in Z; and we deduced from Corollary 2.4 that γ would map all the elements in Z which are not in T upon elements of order 2. But this contradicts the hypothesis (a). Thus it follows that $e(x) = 1$ on Z for every x in Z, and that, therefore, γ is an ordinary isomorphism of Z upon $Z^\gamma = \{z^\gamma\}$.

Case 2. z is not of order 0.

Then Z and Z^γ consists both of $n(z)$ elements. But the subgroup Z^γ of B contains the element z^γ whose order is, by (a), exactly $n(z)$. Hence $Z^\gamma = \{z^\gamma\}$, as we desired to prove.

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[10] See Baer (1), Corollary 2.5, p. 6 and Theorem 3.2, p. 7.

Suppose now that S is a subset of A such that S^γ is a subgroup of B, and that r, t are elements in A such that $r^\gamma(t^\gamma)^{-1}$ is in S^γ. Then there exists one and only one element s in S such that $r^\gamma = s^\gamma t^\gamma$. From condition (b), Lemma (3.1.1) and the group property of S^γ we deduce that $\{s^{e(t)}\}^\gamma = \{s\}^\gamma = \{s^\gamma\} \leq S^\gamma$. Hence $s^{e(t)}$ belongs to S. But $r^\gamma = (s^{e(t)}t)^\gamma$ implies that $r = s^{e(t)}t$; and thus we have shown that $rt^{-1} = s^{e(t)}$ is in S whenever $r^\gamma(t^\gamma)^{-1}$ is in the subgroup S^γ of B. In particular S is a subgroup of A, if S^γ is a subgroup of B. Combining these results with the facts (ii′) and (iii′) verified before, we see now that γ preserves cosets.

Remark. The impossibility of omitting condition (a) in the statement of Theorem 3.1 has actually been pointed out at the end of **2**. The indispensability of condition (b) is made evident by Theorem 1.3.

LEMMA 3.2. *Every integral automorphism e of the group A has the following properties:*

(a) *The set F of fixed elements of e is a normal subgroup of A and the elements of the form x^{e-1} for x in A are in the centralizer*[11] *of F. The elements x^{e-1} are in F, if F contains elements of order* 0.

(b) *If there exist elements of order* 0 *in A which are not fixed elements of e, then e is the inversion: $x^e = x^{-1}$ for every x in A and A is abelian.*

(c) *If e is not the inversion of A, and if A contains elements of order* 0, *then $\{v^{e-1}\} < \{v\}$ for every v in A with $1 < n(v)$.*

(d) $v^{1+e+e^2+\cdots+e^i} = v^{\frac{1}{2}i(i+1)(e-1)+i+1}$ *for v in A and $0 \leq i$, provided F contains elements of order* 0.

Proof. If $u^e = u$ for some u in A, then there exist integers i, j such that $(ux)^i = (ux)^e = u^e x^e = ux^j$ or $x^{e-1} = x^{j-1} = (xu)^{i-1}$. This implies that x^{e-1} permutes both with x and xu and therefore with u. Hence $ux^{e-1} = x^{e-1}u$ or $x^{-1}ux = x^{-e}ux^e = x^{-e}u^e x^e = (x^{-1}ux)^e$. Now it is readily verified that F is a normal subgroup of A and that x^{e-1} belongs to the centralizer of F in A. If F contains an element v of order 0, then v and x^{e-1} generate an abelian subgroup of A. There exists, by Theorem 1.5, an integer k such that $v = v^e = v^k$ and $x^{(e-1)e} = x^{(e-1)k}$. But $k = 1$, since v is of order 0, so that x^{e-1} belongs to F, if F contains elements of order 0.

If u is an element of order 0 such that $u \neq u^e$, then $u^e = u^{-1}$, since e is integral. If x is any element in A, then x and $u^{-1}xu$ are elements of equal

[11] The centralizer of the set S in the group G consists of the elements z in G, satisfying $zs = sz$ for every s in S.

order in A. Hence there exist integers i, j such that $\{x^i\} = \{x\} = \{x^j\}$, and such that

$$u^{-1}x^i u = (u^{-1}xu)^i = (u^{-1}xu)^e = u^{-e}x^e u^e = ux^j u^{-1}$$

or $u^{-2}x^i u^2 = x^j$. The element u^2 transforms, consequently, the cyclic group $\{x\}$ into itself. Since every automorphism of a cyclic group is of finite order, there exists a positive integer n such that $xu^{2n} = u^{2n}x$. The subgroup, generated by x and u^{2n}, is, therefore, abelian. Hence there exists, by Theorem 1.5, an integer k such that $x^e = x^k$ and $u^{-2n} = u^{2ne} = u^{2nk}$. Thus $k = -1$ and $x^e = x^{-1}$ for every x in A. The commutativity of A is a well known consequence of the fact that $x^e = x^{-1}$ is an automorphism of A. This completes the proofs of properties (a) and (b).

If e is not the inversion of A, and if A contains elements of order 0, then it follows from (b) that the elements of order 0 are in F; and it follows from (a) that every x^{e-1} is in the central of F. If x and x^{e-1} generate the same cyclic group, then x is in F and $x^{e-1} = 1$, proving (c).

The identity (d) is certainly true for every element v and $i = 0$. We infer, furthermore, from (a) that $v^{e(e-1)} = v^{e-1}$. Hence we prove by induction that

$$\begin{aligned} v^{1+e+e^2+\cdots+e^i} &= v\, v^{(1+e+\cdots+e^{i-1})e} = v\, v^{\frac{1}{2}(i-1)i(e-1)e} v^{ie} \\ &= v^{1+\frac{1}{2}(i-1)i(e-1)+ie} = v^{\frac{1}{2}i(i+1)(e-1)+i+1} \end{aligned}$$

as we desired to show.

Remark. Since every element of positive order is the product of mutually commutative elements of prime power order, it follows from (c) that $n(v^{e-1})$ is a divisor of $\Pi p_i^{n_i-1}$, if $n(v) = \Pi p_i^{n_i}$ with $0 < n_i$. Thus F contains all the elements with squarefree positive order. The impossibility of proving $F = A$, if A contains elements of order 0 and e is not the inversion, may be seen from the following example.

Let p be a prime number, V a free abelian group of rank $p-1$, $b(1)$, $\cdots, b(p-1)$ a basis of V. Then there exists one and only automorphism α of V which maps $b(i)$ upon $b(i+1)$ for $0 < i < p-1$ and which maps $b(p-1)$ upon $b(1)^{-1}\cdots b(p-1)^{-1}$. This automorphism is of order p. If the integer j is prime to p, then

$$b(i)^{\alpha^j} b(i)^{\alpha^{2j}} \cdots b(i)^{\alpha^{pj}} = 1.$$

Denote by A the group which is obtained by adjoining to V an element w, subject to the relations: $w^{p^2} = 1$ and $w^{-1}vw = v^{\alpha}$ for v in V.

Since the automorphism α of V is of order p, there exists one and only one automorphism β of A such that $v = v^\beta$ for v in V and $w^\beta = w^{1+p}$.

Every element in A is of the form: vw^j for v in V. If j is divisible by p, then $(vw^j)^\beta = vw^{j(1+p)} = vw^j$, since w is of order p^2. If j is prime to p, $v = \prod_i b(i)^{v(i)}$, then

$$\begin{aligned}(vw^j)^{1+p} &= v\, w^j v w^{-j} \cdots w^{pj} v w^{-pj} w^{j+pj} \\ &= v \prod_i (b(i)^{a^{-j}} \cdots b(i)^{a^{-pj}})^{v(i)} w^{j+pj} = v w^{j+pj} = (vw^j)^\beta,\end{aligned}$$

proving that β is an integral automorphism of A which is different from both the identity and the inversion.

THEOREM 3. 3. *Every integral crossed isomorphism γ of a group A, containing elements of order* 0, *upon a group B has the following properties:*

(a) *γ induces an isomorphism in the subgroup J, generated by the elements of order* 0 *in A, if, and only if, $n(x) = 0$ implies that $(x^\gamma)^2$ is not an element in $\{x^4\}^\gamma$.*

(b) *γ is conformal if, and only if, both $n(x) = 0$ and $n(x) = 2^{2+n}$ for $0 \leq n$ imply that $(x^\gamma)^2$ is not an element in $\{x^4\}^\gamma$.*

Remark. $A = J$, if the product of an element of order 0 by an element of positive order is an element of order 0, in particular if A is *abelian.*

Proof. There exists for every element x in A an integral automorphism $e(x)$ of A such that γ is an $e(x)$-isomorphism of A upon the group B. We note furthermore that as a consequence of Theorem 1. 3 subgroups of A are mapped by γ upon subgroups of B.

If $n(x) = 0$, then $x^{e(x)} = x^{\pm 1}$; and we infer from $(x^\gamma)^2 = (x^{1+e(x)})^\gamma$ that $(x^\gamma)^2$ is in $\{x^4\}^\gamma$ if, and only if, x^γ is of order 2.

Suppose now that there exist elements u, v in A such that $n(u) = 0$ and $u^{e(v)} = u^{-1}$. Then we deduce from Lemma 3. 2, (b) that $e(v)$ is the inversion of A and that A is abelian. Consequently it follows from Theorem 1. 5 that every $e(y)$ is either the identity or the inversion, since A contains elements of order 0. Denote by E the subgroup of all the elements t in A such that $e(t) = 1$. Then we deduce from Theorem 2. 1 and Corollary 2. 4 that E is of index 2 in A and that the elements in A, not in E, are mapped upon elements of order 2. But there exist elements of order 0 in A which are not in E, since A is abelian and contains elements of order 0, and since E is of index 2 in A.

It is obvious that the condition (a) is necessary. If the condition (a) is satisfied by γ, then it follows from the second paragraph of this proof that elements of order 0 are mapped by γ upon elements of an order different from 2; and hence it follows from the result obtained in the preceding paragraph of this proof that $n(u) = 0$ implies $u = u^{e(v)}$ for every v in A. Thus $x^{e(y)} = x$ for x in J and y in A, showing that γ induces an isomorphism of J. This completes the proof of (a).

If $n(x) = 2^{2+n}$ for $0 \leq n$, and if $(x^\gamma)^2$ is in $\{x^4\}^\gamma$, then $n(x^\gamma)$ is a divisor of 2^{1+n}. The necessity of the condition of (b) is now readily deduced from the second paragraph of this proof.

Suppose finally that the condition of (b) is satisfied by γ. Then it follows from (a) that every element of order 0 is mapped by γ upon an element of order 0. Thus we have to show that $0 < n(x)$ implies $n(x) = n(x^\gamma)$. Since $1 = 1^\gamma$, we may assume that $n(y) = n(y^\gamma)$ for every element y such that $0 < n(y) < n(x)$. If we denote by k some integer such that $x^{e(x)-1} = x^k$, then we deduce from Lemma 3. 2, (d) that

$$(x^\gamma)^i = (x^{1+e(x)+\cdots+e(x)^{i-1}})^\gamma = (x^{\frac{1}{2}(i-1)ik+i})^\gamma \text{ for } 0 < i;$$

and it follows from Lemma 3. 2, (a) that $x^k = x^{ke(x)} = x^{k(k+1)}$ and consequently $k^2 \equiv 0$ modulo $n(x)$).

Now we distinguish two cases.

Case 1. $n(x)$ is divisible by an odd prime number p.

If we put $i = p$ in the expression for $(x^\gamma)^i$, then we obtain

$$(x^\gamma)^p = (x^{\frac{1}{2}(p-1)pk+p})^\gamma.$$

If q is a prime divisor of $n(x)$, then it follows from $k^2 \equiv 0$ modulo $n(x)$ that q divides k and that therefore $\frac{1}{2}(p-1)k + 1$ is prime to q. Consequently $\frac{1}{2}(p-1)k + 1$ is prime to $n(x)$. Hence x and $x^{\frac{1}{2}(p-1)k+1}$ are of equal order and thus $n(x^{\frac{1}{2}(p-1)pk+p}) = n(x)/p$. Now it follows from our induction hypothesis that

$$n(x^\gamma) = p\, n((x^{\frac{1}{2}(p-1)pk+p})^\gamma) = p\, n(x^{\frac{1}{2}(p-1)pk+p}) = n(x),$$

as was to be shown.

Case 2. $n(x)$ is not divisible by an odd prime.

Then $n(x)$ is a power of 2. If, in particular, $n(x) = 1$ or 2, then $x^{e(x)} = x$, since $e(x)$ is an integral automorphism, showing that $n(x) = n(x^\gamma)$. Thus we may assume that $n(x) = 2^{2+n}$ for $0 \leq n$. Then we infer from

$(x^\gamma)^2 = (x^{1+e(x)})^\gamma = (x^{2+k})^\gamma$, from the evenness of k and from the fact that $(x^\gamma)^2$ is not in $\{x^4\}^\gamma$ that $k \equiv 0$ modulo 4. Hence $n(x^{2+k}) = \frac{1}{2}n(x)$; and now we verify as in Case 1 that $n(x) = n(x^\gamma)$, completing the proof of the theorem.

THEOREM 3.4. *If the group A does not contain elements of order* 0, *if $e(x)$ is, for every element x in A, an integral automorphism of A, and if γ is an $e(x)$-isomorphism of A upon the group B, then the following conditions are necessary and sufficient for conformality of γ:*

(1) *If x is an element of order 2^m with $1 < m$ in A, then $x^{e(x)} \neq x^{-1+4i}$ for some integer i.*

(2) *If $xy = yx$, and if the orders of x and y are relatively prime, then $x = x^{e(y)}$.*

Remark. It is readily seen that the present condition (1) is just the now relevant part of the condition of Theorem 3.3, (b).

Proof. A. Suppose that γ is conformal, that x is an element of order 2^m, $1 < m$, in A and $x^{e(x)} = x^{-1+4i}$ for some integer i. Then x^γ is an element of order 2^m too and $(x^\gamma)^2 = (x^{4i})^\gamma$ is an element of order 2^{m-1}. But the order of x^{4i} is a divisor of 2^{m-2}; and hence the order of $(x^{4i})^\gamma$ is a divisor of 2^{m-2}, a contradiction which proves the necessity of condition (1).

If $xy = yx$, and if the orders of x and y are relatively prime, then xy is of order $n(x)n(y)$. Hence $\{xy\}^\gamma$ is of order $n(x)n(y)$ and contains the element $(xy)^\gamma$ of order $n(x)n(y)$, i. e. $\{xy\}^\gamma$ is a cyclic group. From the commutativity of cyclic groups we infer that

$$(x^{e(y)}y)^\gamma = x^\gamma y^\gamma = y^\gamma x^\gamma = (y^{e(x)}x)^\gamma \text{ or } x^{e(y)}y = y^{e(x)}x.$$

From $xy = yx$ we deduce $y^{e(x)-1} = x^{e(y)-1} = 1$, since the orders of x and y are relatively prime. This completes the proof of the necessity of conditions (1) and (2).

B. Suppose, conversely, that the conditions (1) and (2) are satisfied by the function $e(x)$. If Z is a cyclic subgroup of prime power order p^m, then γ effects a crossed isomorphism of Z upon the subgroup Z^γ of B, as follows from Theorem 1.3. This crossed isomorphism of Z belongs to a function whose values are integral automorphisms of Z which we denote by $e(x)$ also. Denote by T the subgroup of all the elements t in Z such that $e(t) = 1$ (on Z). Then T^γ is a normal subgroup of Z^γ; and it follows from Corollary

1.2 that the function $e(x)$ effects a homomorphism of the group Z^γ into the group of automorphisms of Z which maps exactly T^γ upon 1. Since Z is of order p^m, I^*_Z is essentially the same as the group of rational integers modulo p^m, prime to p; and we infer from classical results [12] and condition (1) that Z^γ/T^γ is a cyclic group of order a power of p. Clearly γ is an ordinary isomorphism of T upon T^γ; and hence the cyclic character of Z^γ is assured, provided $T = Z$. Thus we assume now that $T < Z$. Then there exists an element z in Z such that z^γ generates Z^γ modulo T^γ; and z cannot be an element in T. Since Z is a cyclic group of prime power order, since z is not in $T < Z$, it follows that $T < \{z\}$; and since Z^γ is generated by T^γ and z^γ, it follows that $\{z\}^\gamma = Z^\gamma$ and that therefore z generates Z. Since $e(z)$ may be considered an integer modulo p^m, prime to p, whose order modulo p^m is a power of p, it follows from (1) that the integer $e(z)$ has the form: $e(z) \equiv 1 + p^j z_0$ modulo p^m where z_0 is prime to p, $0 < j < m$, and where $1 < j$ in case $p = 2$ (since $p = 2$ and $T < Z$ imply $1 < m$). The order of $e(z)$ modulo p^m is known [12] to be p^{m-j}; and

$$(z^\gamma)^{p^{m-j}} = (z^{1+e(z)+\cdots+e(z)^{p^{m-j}-1}})^\gamma \text{ is an element in } T^\gamma.$$

But the exponent $1 + e(z) + \cdots + e(z)^{p^{m-j}-1}$ is [12] divisible by p^j, though not by p^{j+1}, showing that $(z^\gamma)^{p^{m-j}}$ is an element of order p^j, since γ is an ordinary isomorphism of T upon T^γ. Hence z^γ is an element of order p^m and Z^γ is a cyclic group of order p^m. Now it is readily seen that

x and x^γ are of equal order, if x is of prime power order.

If x is any element in A, then $x = x_1 \cdots x_k$ where the x_i are mutually commutative elements of mutually relatively prime power orders. It is an immediate consequence of (2) that $x_i^\gamma x_j^\gamma = x_j^\gamma x_i^\gamma$ for $i \neq j$; and now the conformality of γ is easily derived from the facts already established.

Remark. It is easy to construct examples which show that neither of the two conditions (1) and (2) may be omitted, even if we make the additional hypothesis that A is commutative. See Section **9** below.

THEOREM 3.5. *Suppose that there exists an integral crossed isomorphism of the group A upon the group B. Then*

(a) *A contains elements of order* 0 *if, and only if, B contains elements of order* 0; *and*

[12] See Hecke (1), § 13.

(b) *A is a p-group if, and only if, B is a p-group.*[13]

Proof. Denote by γ an integral crossed isomorphism of A upon B. If Z is a cyclic subgroup of A, then it follows from Theorem 1.3 that Z^γ is a subgroup of B; and γ induces an integral crossed isomorphism of Z upon Z^γ. If Z is an infinite cyclic group, then we infer from Theorem 2.1 and Corollary 2.4 that Z^γ contains elements of order 0. If B contains an element u of order 0, then $U = \{u^{\gamma^{-1}}\}$ is a cyclic group which is mapped by γ upon a subgroup U^γ of B which contains the element u of order 0. Hence U^γ and U are infinite groups, showing that $u^{\gamma^{-1}}$ is of order 0. This completes the proof of (a).

Suppose now that A is a p-group and that v is an element in B. Then $V = \{v^{\gamma^{-1}}\}$ is a cyclic group of order a power of p; and the subgroup V^γ of B contains v and is of order a power of p. Hence v is of order a power of p; and B is a p-group. Assume, finally, that B is a p-group and that w is an element in A. Then $\{w\}^\gamma$ is a finite subgroup of B proving that both $\{w\}^\gamma$ and $\{w\}$ are of order a power of p. Hence w is of order a power of p; and this completes the proof.

The following criterion is an immediate consequence of Theorems 3.3, 3.4 and 3.5.

COROLLARY 3.6. *If the group B does not contain elements of order 2, and if the group A either contains elements of order 0 or is a p-group then every integral crossed isomorphism of A upon B is conformal.*

4. Crossed isomorphisms of groups of degree not less than 2. The element u in the group A is said to be *strictly independent of the element v* in A, if $uv = vu$, if $n(u)$ is a multiple of $n(v)$, and if $\{u\} \cap \{v\} = 1$. Likewise *u is strictly independent of the subset S* of A, if u is strictly independent of every element in S. We say that the group A *is of degree not less than* 2, if to every pair of elements u, v in A there exists an element in A which is strictly independent of both u and v. It may be readily verified that an *abelian* group A is of degree not less than 2 if, and only if, one of the following two mutually exclusive conditions is satisfied by A:

(0) A contains at least two independent elements of order 0.

(ϕ) A does not contain elements of order 0; and A contains either no elements of order n or at least two independent ones.

[13] All the elements of a p-group are of order a power of p. Such groups are often called primary groups.

We denote by $C(G)$ the center of the group G.

THEOREM 4.1. *If γ is a coset preserving crossed isomorphism of the group A upon the group B, if the degree of A is not less than 2, and if B contains either elements of order 0 or else there exists to every element x in B an element y in $C(B) \cap C(A)^\gamma$ such that $n(y)$ is a multiple of $n(x)$, then γ is an ordinary isomorphism of A upon B.*

Remark. It is obvious that the first condition is necessary whereas the other two are not. If A happens to be abelian, then the third condition reads as follows:

If B does not contain elements of order 0, then there exists to every element x in B an element of order $n(x)$ in $C(B)$.

This condition is clearly satisfied, if B is also abelian; and either of these conditions is necessary.

Proof. There exists to every element x in A an endomorphism $e(x)$ of A such that γ is an $e(x)$-isomorphism of A upon B; and it is a consequence of Theorem 3.1 that every $e(x)$ is an integral automorphism of A. If B contains elements of order 0, then A contains elements of order 0 too; and there exists to every element x in A an element y of order 0 in A such that $xy = yx$. Consequently every element in A is a quotient of elements of order 0 in A; and it follows from Theorem 3.3, (a) that γ is an ordinary isomorphism of A upon B. If A and B do not contain elements of order 0, then we proceed as follows: if x is an element in A, and y an element in $C(B)^{\gamma^{-1}}$, then there exists an element z in A which is strictly independent of both x and y. Since y^γ is in $C(B)$, we find $(z^{e(y)}y)^\gamma = z^\gamma y^\gamma = y^\gamma z^\gamma = (y^{e(z)}z)^\gamma$ and therefore $z^{e(y)}y = y^{e(z)}z$. But $e(v)$ is an integral automorphism and z is strictly independent of y. Hence $z^{e(y)} = z$. The elements x and z form a basis of an abelian subgroup of A and $n(z)$ is a multiple of $n(x)$. Hence it follows from Theorem 1.5 that $x^{e(y)} = x$ is a consequence of $z^{e(y)} = z$; and thus we have shown that $e(y) = 1$ for y in $C(B)^{\gamma^{-1}}$. To every element u in A there exists an element v of order $n(u)$ in $C(A) \cap C(B)^{\gamma^{-1}}$. From $e(v) = 1$ and Corollary 1.2, (iv) we deduce now that $v^\gamma = (w^\gamma)^{-1}v^\gamma w^\gamma = (w^{-1}v^{e(w)}w)^\gamma = (v^{e(w)})^\gamma$ for every w in A and that therefore $v = v^{e(w)}$. But u and v generate an abelian subgroup and $e(w)$ is an integral automorphism. Hence it follows from Theorem 1.5 that $u = u^{e(w)}$ for every u and w in A, showing that γ is an ordinary isomorphism of A upon B.

COROLLARY 4.2. *Every coset preserving crossed isomorphism of an abelian group upon a group of degree not less than 2 is an ordinary isomorphism.*

Proof. If A is an abelian group and if γ is a coset preserving crossed isomorphism of A upon the group B of degree not less than 2, then A is of degree not less than 2, since γ is, by Theorem 3.1, conformal. Hence we deduce from Theorem 4.1 that γ is an ordinary isomorphism, if A and/or B contains elements of order 0. Thus we assume now that neither A nor B contains elements of order 0. If u and v are elements in A, then there exist elements r, s in A such that $\{u, v\} = \{r, s\}$ and such that s is strictly independent of r. There exists an element t in A such that t^γ is strictly independent of both r^γ and s^γ. Since γ is conformal, and since A and B do not contain elements of order 0, we may assume without loss of generality that $n(t) = n(s)$. Clearly $\{r, s, t\}$ is a subgroup of degree not less than 2 of A; and t^γ is in the center of $\{r, s, t\}^\gamma$. Hence it follows from Theorem 4.1 that γ is an ordinary isomorphism of the abelian group $\{r, s, t\}$ upon $\{r, s, t\}^\gamma$. From $\{u, v\} = \{r, s\}$ we deduce now that $u^\gamma v^\gamma = (uv)^\gamma$; and γ is consequently an ordinary isomorphism.

LEMMA 4.3. *If ϕ is an $a(x)$- and γ a $b(x)$-isomorphism of the group A upon the group B, then*

$$\phi\gamma^{-1}b(x^{\phi\gamma^{-1}}) = a(x)\phi\gamma^{-1} \quad \text{for } x \text{ in } A$$

is a necessary and sufficient condition for $\phi\gamma^{-1}$ to be an automorphism of A. If, in particular, either $\phi\gamma^{-1}$ is an integral automorphism or ϕ and γ are integral crossed isomorphisms and $\phi\gamma^{-1}$ is an automorphism, then

$$a(x^{\phi^{-1}}) = b(x^{\gamma^{-1}}) \quad \text{for } x \text{ in } B.$$

Proof. If x and y are elements in A, then

$$(xy)^{\phi\gamma^{-1}} = (x^{a(y)^{-1}\phi}y^\phi)^{\gamma^{-1}} = x^{a(y)^{-1}\phi\gamma^{-1}b(y^{\phi\gamma^{-1}})}y^{\phi\gamma^{-1}};$$

and from this identity our contentions are readily deduced.

THEOREM 4.4. *If A is a group of degree not less than 2 and if ϕ and γ are integral crossed isomorphisms of the group A upon some group B, then the following two properties of ϕ and γ are equivalent:*

(i) *$\phi\gamma^{-1}$ is an integral automorphism of A.*

(ii) *$S^\phi = S^\gamma$ for every cyclic subgroup S of A.*

Proof. If $\phi\gamma^{-1}$ is an integral automorphism of A, and if S is a subgroup of A, then $S^{\phi\gamma^{-1}} = S$ and therefore $S^{\phi} = S^{\gamma}$. Hence (ii) is a consequence of (i).

Suppose now, conversely, that (ii) is satisfied by ϕ and γ. Then $\kappa = \phi\gamma^{-1}$ is a 1:1 correspondence between the elements in A such that $A^{\kappa} = A$ and such that there exists to every element x in A an integer $k(x)$ satisfying:

x and $x^{k(x)} = x^{\kappa}$ generate the same cyclic subgroup of A.

There exist, furthermore, uniquely determined functions $a(x)$ and $b(x)$ such that ϕ is an $a(x)$- and γ a $b(x)$-isomorphism of A upon B. We note that both $a(x)$ and $b(x)$ are, for every x in A, integral automorphisms of A. Consequently $d(x) = b(x^{k(x)})a(x)^{-1}$ is a well defined integral automorphism of A for every x in A.

If z and t are elements in A such that $zt = tz$, then

$$\begin{aligned}(z^{a(t)k(z^{a(t)}t)}t^{k(z^{a(t)}t)})^{\gamma} &= ((z^{a(t)}t)^{k(z^{a(t)}t)})^{\gamma} = (z^{a(t)}t)^{\phi} \\ &= z^{\phi}t^{\phi} = z^{k(z)\gamma}t^{k(t)\gamma} = (z^{k(z)b(t^{k(t)})}t^{k(t)})^{\gamma};\end{aligned}$$

and this implies that

(4.4.0) $\qquad z^{a(t)k(z^{a(t)}t)}t^{k(z^{a(t)}t)} = z^{k(z)b(t^{k(t)})}t^{k(t)}$ whenever $zt = tz$.

If t is strictly independent of z, then it is possible to apply this equation; and, using the integrality of $a(x)$ and $b(x)$, we obtain the following result:

(4.4.1) If t is strictly independent of z, then

$$k(t) \equiv k(z^{a(t)}t) \text{ modulo } n(t) \quad \text{and} \quad z^{a(t)k(t)} = z^{k(z)b(t^{k(t)})}.$$

But $a(t)$ is an integral automorphism and $k(t)$ an integer. Hence the following statement may be derived from (4.4.1)

(4.4.2) If t is strictly independent of z, then $z^{k(t)} = z^{k(z)d(t)}$.

If x and y are any two elements in A, then there exists an element t in A which is strictly independent of x and of y. Since x, t as well as y, t generate abelian subgroups of A, and since $d(t)$ is an integral automorphism of A, we infer from Theorem 1.5 the existence of integers i, j such that $x^{d(t)} = x^{i}$, $t^{i} = t^{d(t)} = t^{j}$ and $y^{j} = y^{d(t)}$; and this implies, in particular, that $x^{d(t)} = x^{j}$ too. Thus we deduce from (4.4.2) that $x^{k(t)} = x^{k(x)j}$ and $y^{k(t)} = y^{k(y)j}$ and this may be restated as follows:

if x and y are elements in A, then there exists an integer h such that

$$x^{k(x)} = x^h, \quad y^h = y^{k(y)}.$$

If $x(1), \cdots, x(s)$ are a finite number of elements in A, then there exists an element y in A whose order is a common multiple of all the orders $n(x(i))$. Hence there exist integers $h(i)$ such that $x(i)^{k(x(i))} = x(i)^{h(i)}$ and $y^{k(y)} = y^{h(i)}$. This shows the validity of the following statement:

if $x(1), \cdots, x(s)$ are a finite number of elements in A, then there exists an integer h such that $x(i)^{k(x(i))} = x(i)^h$ for $i = 1, \cdots, s$.

If $zt = tz$ for t and z in A, then there exists an integer r such that $z^{k(z)} = z^r$, $(z^{a(t)}t)^{k(z^{a(t)}t)} = (z^{a(t)}t)^r$ and $t^{k(t)} = t^r$. Hence we deduce from (4.4.0) that $z^{a(t)r}t^r = z^{rb(t^{k(t)})}t^r$ or $z^{a(t)} = z^{b(t^{k(t)})}$. Thus we have shown:

(4.4.3) $$z = z^{d(t)}, \text{ if } zt = tz.$$

If x and y are any two elements in A, then there exists an element z in A which is strictly independent of both x and y. From (4.4.3) we infer that $z^{d(y)} = z$. Since z and x generate an abelian subgroup of A, and since the order of z is a multiple of the order of A, we infer from Theorem 1.5 and the integrality of $d(y)$ that $x^{d(y)} = x$. Thus we have shown that $d(y) = 1$ for every y in A; or what amounts to the same thing:

$$a(y) = b(y^{k(y)}) \text{ for every } y \text{ in } A.$$

If x and y are elements in A, then it follows from what has been shown that

$$x^{\phi\gamma^{-1}b(y^{\phi\gamma^{-1}})} = x^{k(x)b(y^{k(y)})} = x^{a(y)k(x)} = x^{a(y)k(x^{a(y)})} = x^{a(y)\phi\gamma^{-1}}$$

or $\phi\gamma^{-1}b(y^{\phi\gamma^{-1}}) = a(y)\phi\gamma^{-1}$ for every y in A. Hence it follows from Lemma 4.3 that $\phi\gamma^{-1}$ is an automorphism of A; and the integrality of $\phi\gamma^{-1}$ is now immediately obvious.

A *projectivity* [14] *of the group A upon the group B* is an isomorphism of the partially ordered set of all the subgroups of A upon the partially ordered set of all the subgroups of B. In other words: a projectivity is a 1:1 and monotone increasing mapping of the system of subgroups of A upon the system of subgroups of B. A projectivity π of the group A upon the group B is termed *index preserving*, if it has the following property:

(Z) If Z is a cyclic subgroup of A, and if S is a subgroup of Z, then the index of S in Z and the index of S^π in Z^π are equal.

[14] For the concepts concerning projectivities, see Baer (1) and Beaumont (1), (2).

The projectivity π is said to be *strictly index preserving*, if it meets the stronger requirement:

(S) If T is a subgroup of A and S a subgroup of T, then the index of S in T and the index of S^π in T^π are equal.

The $1:1$ correspondence ϕ between the elements in the group A and the elements in the group B $(A^\phi = B)$ induces a projectivity, if it satisfies the following condition:

S is a subgroup of A if, and only if, S^ϕ is a subgroup of B.

It is well known [15] that every projectivity of the group A upon the group B maps cyclic subgroups upon cyclic subgroups which clearly are of the same order, if the projectivity is induced by a $1:1$ correspondence. This shows the conformality of $1:1$ correspondences between groups which induce projectivities. It should be noted that in many instances conformality of a projectivity will suffice to assure index preservation in the strict sense.

THEOREM 4. 5. *The* $1:1$ *correspondence* γ *mapping the group* A *of degree not less than* 2 *upon the group* B *is an integral crossed isomorphism of* A *upon* B *if, and only if,*

$$\{uv^{-1}\} = \{(u^\gamma(v^\gamma)^{-1})^{\gamma^{-1}}\} \text{ for } u \text{ and } v \text{ in } A.$$

Remark. The necessity of the condition is independent of the hypothesis that A is of degree not less than 2.

Proof. A. If γ is an integral crossed isomorphism of A upon B, then there exists to every element x in A an integral automorphism $e(x)$ of A such that γ is an $e(x)$-isomorphism. If u and v are elements in A, and if $w = uv^{-1}$, then $u^\gamma = (wv)^\gamma = (w^{e(v)^{-1}})^\gamma v^\gamma$ or $(u^\gamma(v^\gamma)^{-1})^{\gamma^{-1}} = w^{e(v)^{-1}} = (uv^{-1})^{e(v)^{-1}}$, proving the necessity of the condition, since $e(v)^{-1}$ is an integral automorphism of A.

B. If, conversely, the condition is satisfied by γ, and if u and v are elements in the subgroup S of A, then $\{uv^{-1}\} \leq S$ implies $\{uv^{-1}\}^\gamma \leq S^\gamma$. But $(u^\gamma(v^\gamma)^{-1})^{\gamma^{-1}}$ is, by our hypothesis, an element in $\{uv^{-1}\}$, showing that $u^\gamma(v^\gamma)^{-1}$ belongs to S^γ; and consequently every subgroup of A is mapped by γ upon a subgroup of B.

If u and v are any elements in A, then it follows from our hypothesis that $\{((uv)^\gamma(v^\gamma)^{-1})^{\gamma^{-1}}\} = \{(uv)v^{-1}\} = \{u\}$; and hence there exists an in-

[15] See footnote [10] above.

teger (u, v) such that $\{u\} = \{u^{(u,v)}\}$ and $u^{(u,v)} = ((uv)^{\gamma}(v^{\gamma})^{-1})^{\gamma^{-1}}$. This equation we restate in the following handier form:

(4.5.1) $$(uv)^{\gamma} = (u^{(u,v)})^{\gamma} v^{\gamma} \text{ for } u \text{ and } v \text{ in } A.$$

If x, y, z are elements in A, then we infer from the associative law in A that

$$((xy)^{(xy,z)})^{\gamma} z^{\gamma} = (xyz)^{\gamma} = (x^{(x,yz)})^{\gamma}(yz)^{\gamma} = (x^{(x,yz)})^{\gamma}(y^{(y,z)})^{\gamma} z^{\gamma} \quad \text{or}$$

(4.5.2) $$(xy)^{(xy,z)\gamma} = x^{(x,yz)\gamma} y^{(y,z)\gamma} \text{ for } x, y, z \text{ in } A.$$

If X and Y are the cyclic groups, generated by x and y respectively, then X^{γ} and Y^{γ} are subgroups of B, as has been shown in the first paragraph of part B of this proof. If $xy = yx$, and if the cross cut of X and Y is 1, in short: if the elements x and y in A are independent elements in A, then X^{γ} and Y^{γ} are subgroups of B whose cross cut is 1; and we deduce from (4.5.1) and (4.5.2) that

(4.5.3) $$\begin{aligned} x^{(x,yz)\gamma} y^{(y,z)\gamma} &= (x^{(xy,z)} y^{(xy,z)})^{\gamma} \\ &= x^{(xy,z)(x^{(xy,z)},y^{(xy,z)})\gamma} y^{(xy,z)\gamma} \end{aligned}$$

and that therefore $y^{(y,z)\gamma} = y^{(xy,z)\gamma}$ and consequently $y^{(y,z)} = y^{(xy,z)}$ or

$$(y, z) \equiv (xy, z) \text{ modulo } n(y).$$

Since our hypotheses are symmetric in x and y, we infer now that

$$(x, z) \equiv (yx, z) \equiv (xy, z) \text{ modulo } n(x),$$

since $xy = yx$. Thus we have shown:

If x and y are independent elements in A, then there exists to every element z in A an integer i such that $x^{(x,z)} = x^i$ and $y^{(y,z)} = y^i$.

If u, v, w are elements in A, then there exists an element t in A which is strictly independent of both u and v. From the result just obtained we deduce the existence of integers i, j such that $u^{(u,w)} = u^i$, $t^i = t^{(t,w)} = t^j$, $v^j = v^{(v,w)}$. Since $n(t)$ is a multiple of $n(u)$, it follows that $u^i = u^j$; and thus we have shown:

(4.5.4) If x, y, z are elements in A, then there exists an integer i such that $x^{(x,z)} = x^i$ and $y^{(y,z)} = y^i$.

We *define* a function $d(x)$ for x in A by the equation:

$$y^{d(x)} = y^{(y,x)} \text{ for every } y \text{ in } A.$$

Then y and $y^{d(x)}$ generate the same cyclic subgroup of A; and the Lemma 4. 5. 4 may be restated as follows:

(4. 5. 4′) If x, y, z are elements in A then there exists an integer i such that $x^{d(z)} = x^i$ and $y^{d(z)} = y^i$.

If x, y, z are elements in A, then there exists an element t in A which is strictly independent of both x and y. Substituting in (4. 5. 3) the element t for the element x we obtain, as before: $t^{d(yz)} = t^{d(z)d(y^{d(z)})}$; and now one readily deduces from (4. 5. 4′) and the fact that $n(t)$ is a multiple of $n(x)$ the following equation: $x^{d(yz)} = x^{d(z)d(y^{d(z)})}$. Consequently we deduce from (4. 5. 1) and (4. 5. 2) that

$$(xy)^{d(z)\gamma} = x^{d(z)d(y^{d(z)})\gamma} y^{d(z)\gamma} = (x^{d(z)}y^{d(z)})^{\gamma} \text{ or } (xy)^{d(z)} = x^{d(z)}y^{d(z)}.$$

Thus we have shown that every $d(z)$ is an integral automorphism of A; and hence $e(x) = d(x)^{-1}$ is, for every x, an integral automorphism of A. Now we infer from (4. 5. 1) that

$$u^{\gamma}v^{\gamma} = u^{e(v)d(v)\gamma}v^{\gamma} = (u^{e(v)}v)^{\gamma},$$

proving that γ is an $e(x)$-isomorphism and therefore an integral crossed isomorphism of A upon B.

Corollary 4. 6. *The* $1:1$ *correspondence* γ *of the elements in the group* A *of degree not less than* 2 *upon the group* $B = A^{\gamma}$ *is a coset preserving crossed isomorphism of* A *upon* B *if, and only if,*

$$\{uv^{-1}\}^{\gamma} = \{u^{\gamma}(v^{\gamma})^{-1}\} \textit{ for } u \textit{ and } v \textit{ in } A.$$

Remark. This condition is satisfied by every coset preserving crossed isomorphism of every group A.

Proof. A. If γ is a coset preserving crossed isomorphism of A upon B, then it follows from Lemma (3. 1. 1) that $\{z^{\gamma}\} = \{z\}^{\gamma}$ for every element z in A. Since γ is integral by Theorem 3. 1, we infer now from Theorem 4. 5 that

$$\{uv^{-1}\}^{\gamma} = \{(u^{\gamma}(v^{\gamma})^{-1})^{\gamma^{-1}}\}^{\gamma} = \{u^{\gamma}(v^{\gamma})^{-1}\},$$

proving the necessity of our condition.

B. Suppose now that our condition is satisfied by γ. Then we infer that $1^{\gamma} = 1$ and that therefore $\{x\}^{\gamma} = \{x^{\gamma}\}$ for x in A, showing that γ is conformal. Hence $\{y^{\gamma^{-1}}\} = \{y\}^{\gamma^{-1}}$ for y in B and consequently

$$\{uv^{-1}\} = \{uv^{-1}\}^{\gamma\gamma^{-1}} = \{u^{\gamma}(v^{\gamma})^{-1}\}^{\gamma^{-1}} = \{(u^{\gamma}(v^{\gamma})^{-1})^{\gamma^{-1}}\} \text{ for } u, v \text{ in } A;$$

and thus we deduce from Theorem 4. 5 that γ is an integral crossed isomorphism of A upon B. It follows now from Theorem 3. 1 that γ is a coset preserving crossed isomorphism.

Remark. Theorem 4. 5 and Corollary 4. 6 fail to be true, if we omit the hypothesis that the group A is of degree not less than 2. This may be seen from the following example: If A is a cyclic group of order a prime number $p \neq 2, 3$, then every permutation of the elements in A which leaves 1 invariant satisfies the conditions of Theorem 4. 5 and Corollary 4. 6; and the number of such permutations is $(p-1)!$. The number of crossed isomorphisms of A upon itself is exactly $p - 1 < (p-1)!$, showing that some of the transformations, meeting the requirements of Theorem 4. 5 and Corollary 4. 6, fail to be crossed isomorphisms.

COROLLARY 4. 7. *Suppose that A and B are both of degree not less than 2, and that the $1:1$ correspondence γ maps A upon B. Then γ is a coset preserving crossed isomorphism of A upon B if, and only if, γ^{-1} is a coset preserving crossed isomorphism of B upon A.*

This is an almost immediate consequence of Corollary 4. 6. Cp. in this context Theorem 1. 4′.

The following theorem serves to show the impossibility of substituting products for quotients in Corollary 4. 6.

If A is an abelian group such that there exists to every pair of elements u and v in A an element w in A strictly independent of the subgroup $\{u, v\}$ of A, if γ is a $1:1$ correspondence mapping A upon the group B, and if $\{xy\}^{\gamma} = \{x^{\gamma}y^{\gamma}\}$ for x and y in A, then γ is an ordinary isomorphism of A upon B.

Remark. It will be shown in **7** that such a group A may admit of coset preserving crossed isomorphisms which are not ordinary isomorphisms.

Proof. If x and y are elements in A, then there exists an integer (x, y) such that $\{xy\} = \{(xy)^{(x,y)}\}$ and such that $(xy)^{(x,y)\gamma} = x^{\gamma}y^{\gamma}$. If x, y, z are elements in A, then we deduce from the associativity of multiplication in B that

$$\begin{aligned}((xy)^{(x,y)}z)^{((xy)^{(x,y)},z)\gamma} &= (xy)^{(x,y)\gamma}z^{\gamma} = x^{\gamma}y^{\gamma}z^{\gamma} \\ &= x^{\gamma}(yz)^{(y,z)\gamma} = (x(yz)^{(y,z)})^{(x,(yz)^{(y,z)})\gamma};\end{aligned}$$

since γ is a $1:1$ correspondence, it follows now that

(a) $$((xy)^{(x,y)}z)^{((xy)^{(x,y)},z)} = (x(yz)^{(y,z)})^{(x,(yz)^{(y,z)})}.$$

If the element v in A is strictly independent of the element u in A, then there exists an element w in A strictly independent of $\{u, v\}$. The elements u, v, w form a basis of $\{u, v, w\}$. Hence we may deduce from (a) and the commutativity of A that

$$((uv)^{(u,v)}, w) \equiv (v, w)(u, (vw)^{(v,w)}) \text{ modulo } n(w),$$
$$(u, v)((uv)^{(u,v)}, w) \equiv (v, w)(u, (vw)^{(v,w)}) \text{ modulo } n(v),$$
$$(u, v)((uv)^{(u,v)}, w) \equiv (u, (vw)^{(v,w)}) \text{ modulo } n(u).$$

Combining the first two congruences and considering that $n(w)$ is a multiple of $n(v)$, we find that $(u, v) \equiv 1$ modulo $n(v)$.

If x and y are elements in A, then there exists an element z in A strictly independent of $\{x, y\}$. Then z is strictly independent of y and of $(xy)^{(x,y)}$; and $(yz)^{(y,z)}$ is strictly independent of x. Hence it follows from (a), the commutativity of A and from the result of the preceding paragraph of the proof, that $(xy)^{(x,y)}z = xyz$ and that therefore $xy = (xy)^{(x,y)}$, proving the multiplicativity of γ.

5. Crossed isomorphisms of subgroups derived from a projectivity of the group. It has been pointed out before that coset preserving crossed isomorphisms induce projectivities which are index preserving in the strict sense. Theorem 4.4 shows that integral crossed isomorphisms which have the same effect on the system of subgroups are not very much different. The following two sections are devoted to the converse problem: given a projectivity, to find a crossed isomorphism which induces the given projectivity in the system of subgroups.

Projectivities are known [16] to map cyclic subgroups upon cyclic subgroups, although in general they do not preserve the orders of cyclic subgroups. Accordingly we shall call a projectivity *conformal*, if it maps every cyclic subgroup upon a cyclic group of equal order. Clearly index preserving projectivities are conformal, but conformal projectivities need not be index preserving.

There exist projectivities which are strictly index preserving and which map an abelian group upon a not commutative group.[17] Thus concepts like "commutative group" and "strictly independent element" are not invariant under conformal projectivities. This leads us to the following definitions which are invariant substitutes for these concepts.

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[16] See footnote [10] above. [17] See Baer (1), Beaumont (1), (2), Rottlaender (1).

The cyclic subgroup Z of the group A is said to be strictly independent of the subgroup S of A, if the following conditions are satisfied by Z and S:

(5. i) *If X is a subgroup of Z and $Z \leq XS$, then $X = Z$.*

(5. ii) *If V is a cyclic subgroup of S and U a subgroup of V, and if $V \leq ZU$, then $U = V$.*

(5. iii) *If T is a cyclic subgroup of ZS such that $T \nleq YS$ for every proper subgroup Y of Z, then $T \cap S = 1$.*

It is an immediate consequence of (5. i) and (5. iii) that $Z \cap S = 1$.

The concept of strict independence is invariant under projectivities, since projectivities map cyclic groups upon cyclic groups.

THEOREM 5. 1. *The element u in the group A is strictly independent of the element v in A if, and only if, $uv = vu$ and the cyclic subgroup $\{u\}$ is strictly independent of the cyclic subgroup $\{v\}$.*

Remark. This shows that strictly independent elements generate strictly independent cyclic subgroups, although strictly independent cyclic subgroups need not be generated by strictly independent elements.

Proof. If the element u in A is strictly independent of the element v in A, then the elements u, v form a basis of the abelian subgroup $\{u, v\}$. Every element in this subgroup may therefore be represented in one and essentially only one fashion in the form $t = u^i v^j$. Thus conditions (5. i) and (5. ii) are satisfied by $Z = \{u\}$ and $S = \{v\}$. If $T = \{u^i v^j\}$ and $T \cap S \neq 1$, then there exists an integer k such that $(u^i v^j)^k = v^{jk} \neq 1$. Clearly $n(u)$ is a divisor of ik. Since $n(v)$ is a divisor of $n(u)$, it follows that $n(v)$ is a divisor of ik. But $n(v)$ is not a divisor of k. Hence $\{u^i\} < \{u\}$ and $T \leq \{u^i\}S$. Thus (5. iii) is satisfied by Z and S.

If, conversely, $uv = vu$ and $\{u\}$ is strictly independent of $\{v\}$, then $\{u\} \cap \{v\} = 1$ so that the elements u, v form a basis of the abelian group $\{u, v\}$. If $\{u^i\} < \{u\}$, then the element uv is not contained in $\{u^i\}\{v\}$. Hence it follows from (5. iii) that $(uv)^{n(u)} = v^{n(u)} = 1$, proving that $n(v)$ is a divisor of $n(u)$ and that therefore the element u is strictly independent of the element v in A.

The element u in the group A is termed *an element domineering the subgroup S of A*, if $\{su\}$ is strictly independent of S for every s in S.

Whether or not every strictly independent element is a domineering element, the author has not been able to decide. This can be shown to be true in a fairly large class of groups, the *R-groups* which meet the following requirement:

(R) *If the orders of the elements $x_1, \cdots, x_k$ in A are divisors of n, then the order of their product is a divisor of n.*

Examples of R-groups are the abelian groups and the regular p-groups of P. Hall.[18] It is readily seen that conformal projectivities map R-groups upon R-groups.

THEOREM 5. 2. *If the element u in the R-group A is strictly independent of the subgroup S of A, then u domineers S.*

Remark. If u is strictly independent of the subgroup S of A, then it is not difficult to prove the following fact: su, for s in S, is strictly independent of S if, and only if, s is in the center of S.

Proof. For s an element in S, put $Z = \{su\}$. If $X \leq Z$, then $X = \{(su)^i\}$ for i a divisor of $n(u) = n(su)$. If $Z \leq XS$, then u is in $XS = \{u^i\}S$ and consequently $\{u\} = \{u^i\}$ and $X = Z$, since $(su)^i = s^i u^i$. If, furthermore, v is an element in S, $V = \{v\}$ and U a subgroup of V, then $U = \{v^j\}$ for j a divisor of $n(v)$. If $V \leq ZU$, then there exist integers $h(1), \cdots, h(k)$ and $m(1), \cdots, m(k)$ such that

$$\begin{aligned} v &= v^{jh(1)}(su)^{m(1)} \cdots v^{jh(k)}(su)^{m(k)} \\ &= v^{jh(1)} s^{m(1)} v^{jh(2)} s^{-m(1)} s^{m(1)+m(2)} v^{jh(3)} s^{-(m(1)+m(2))} \cdots \\ &\qquad \cdots s^{m(1)+\cdots+m(k-1)} v^{jh(k)} s^{-(m(1)+\cdots+m(k-1))} (su)^{m(1)+\cdots+m(k)} \\ &= s' s^{m(1)+\cdots+m(k)} u^{m(1)+\cdots+m(k)}. \end{aligned}$$

Since A is an R-group, and since s' is a product of transforms of powers of v^j, we have: $n(s')$ is a divisor of $n(v^j)$. Since s', s and v are in S, it follows that $u^{m(1)+\cdots+m(k)} = 1$. Since u is strictly independent of s, it follows now that $s^{m(1)+\cdots+m(k)} = 1$. Hence $v = s'$ and $n(v)$ is a divisor of $n(v^j)$. Consequently $\{v\} = \{v^j\}$; and thus we have shown that conditions (5. i) and (5. ii) are satisfied by S and Z.

If the cyclic group $T \leq ZS$, then there exists an integer i and an element s'' in S such that $T = \{s''u^i\}$, since $ZS = \{u\}S$. If $T \cap S \neq 1$, then it follows from Theorem 5. 1 that $\{u^i\} < \{u\}$, since the element u is strictly independent of S and therefore of $\{s''\}$. But $s''u^i = s''s^{-i}(su)^i$ and $\{(su)^i\} < \{su\}$, since the order of u is a multiple of $n(s)$ and since $su = us$. Thus $T \cap S \neq 1$ implies $T \leqq \{(su)^i\}S$ and $\{(su)^i\} < Z$. Thus condition (5. iii) is satisfied by S and Z; and we have shown that Z is strictly independent of S. Hence u domineers S.

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[18] Hall (1), Theorem 4. 26, p. 76.

The group A is called an *A-group*, if the following condition is satisfied by the subgroups of A:

(A) *If V is a cyclic subgroup of A and u an element in A such that $\{u\}$ is strictly independent of V, then the following two properties of cyclic subgroups Z of $\{u\}V$ imply each other.*

(A') *There exists an element v such that $V=\{v\}$ and $Z=\{vu\}$.*

(A'') *If X is a subgroup of $\{u\}$ such that $Z \leq XV$, then $X=\{u\}$; and if Y is a subgroup of V such that $Z \leq \{u\}Y$, then $Y=V$.*

The following two theorems may serve as a justification of this definition.

THEOREM 5. 3. *Every abelian group is an A-group.*

Proof. The element u in the abelian group A is, by Theorem 5. 1, strictly independent of the cyclic subgroup V of A if, and only if, the cyclic subgroup $\{u\}$ of A is strictly independent of V. If u is strictly independent of V, if x generates V and u^i generates $\{u\}$, then there exists an element y such that $V=\{y\}$ and $\{xu^i\}=\{yu\}$. If v is an element in V, W a subgroup of V such that $\{vu^i\} \leq W\{u\}$, then v is in W; and if K is a subgroup of $\{u\}$ such that $\{vu^i\} \leq VK$, then u^i is in K; and now it is clear how to complete the proof.

THEOREM 5. 4. *If A is an A-group, and if there exists a conformal projectivity of A upon the group B, then B is an A-group.*

Remark. From the existence of conformal projectivities mapping abelian groups upon non abelian groups we infer now the existence of non abelian A-groups.

Proof. If the cyclic subgroup $U=\{u\}$ of the A-group A is strictly independent of the cyclic subgroup V of A, if $V=\{v'\}=\{v''\}$ and $w=\{v'u\}$ $=\{v''u\}$, then W meets the requirement (A'); and it follows from (A'') that $W \leq VX$ and $X \leq U$ imply $X=U$. Hence we deduce from (5. iii) that $W \frown V=1$. But $v'v''^{-1}=v'u(v''u)^{-1}$ is in the cross cut of W and V. Hence $v'=v''$. This shows that the cyclic subgroups Z of $\{u\}V$, meeting the requirement (A''), may be represented in one and only one way in the form: $Z=\{vu\}$ with $V=\{v\}$; and that the number of subgroups Z of $\{u\}V$, meeting the requirement (A''), is exactly $n^*(V)$, the finite number of different elements in V which may serve as generators of V.

There exists a conformal projectivity π of A upon B. Suppose that the cyclic subgroup $\{b\}$ of B is strictly independent of the cyclic subgroup P of B.

Since strict independence of cyclic subgroups is invariant under projectivities, it follows that $U = \{b\}^{\pi^{-1}}$ is strictly independent of $V = P^{\pi^{-1}}$. Since π is conformal, it follows that V and P are cyclic groups of equal order and that therefore $n^*(V) = n^*(P)$. Projectivities map triplets of cyclic subgroups Z, U, V which meet the requirement (A'') upon triplets of cyclic subgroups satisfying (A''). Hence it follows from the result of the first paragraph of this proof that $n^*(P)$ is the number of cyclic subgroups Q of $\{b\}P$ which meet the requirement (A'').

If $P = \{g\}$, $X \leq \{b\}$ and $\{gb\} \leq XP$, then $\{b\} \leq XP$; and it follows from (5. i) that $X = \{b\}$. Hence we deduce from (5. iii) that $\{gb\} \cap P = 1$. Thus we show, as in the first paragraph of this proof, that $P = \{g'\} = \{g''\}$ and $\{g'b\} = \{g''b\}$ imply $g' = g''$. If $P = \{g\}$, $Y \leq P$ and $\{gb\} \leq \{b\}Y$, then $P \leq \{b\}Y$; and it follows from (5. ii) that $P = Y$. Hence there exist exactly $n^*(P)$ cyclic subgroups of $\{b\}P$ which meet the requirement (A') and they all meet the requirement (A''). But the number of the latter subgroups has been shown in the second paragraph of the proof to be exactly $n^*(P)$ too. Hence we infer from the finiteness of the number $n^*(P)$ that exactly the same cyclic subgroups meet the requirement (A') as satisfy condition (A''); and this shows that B is an A-group too.

The following lemmas will be needed both for the proof of the main theorem of this section and for some applications in the next section.

LEMMA 5. 5. *If π is a conformal projectivity of the A-group A upon the group B, if s and t are elements in A such that $\{t\}$ is strictly independent of $\{s\}$, and if $\{t'\} = \{t\}^{\pi}$, then there exists one and only one solution $s' = g(s; t, t') = g(s; t, t', \pi)$ of the equations:*[10]

$$(5. 5. 1) \qquad \{s'\} = \{s\}^{\pi} \quad \text{and} \quad \{s't'^{-1}\} = \{st^{-1}\}^{\pi}.$$

Remark. $g(1; t, t', \pi) = 1$, if $\{t'\} = \{t\}^{\pi}$.

Proof. It is a consequence of Theorem 5. 4 that B is an A-group too. $\{st^{-1}\}$ meets (A') with regard to $\{s\}$ and t^{-1}. Hence $\{st^{-1}\}$ meets (A'') with regard to $\{s\}$ and $\{t^{-1}\}$. Consequently (A'') is satisfied by $\{st^{-1}\}^{\pi}$ with regard to $\{s\}^{\pi}$ and $\{t^{-1}\}^{\pi} = \{t'^{-1}\}$. Hence (A') is satisfied by $\{st^{-1}\}^{\pi}$ with regard to $\{s\}^{\pi}$ and t'^{-1}. This proves the existence of an element s', satisfying (5. 5. 1); and its unicity is verified as usual (see the first or second paragraph of the proof of Theorem 5. 4).

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[10] In the notation, used by Baer (1), p. 17, we have: $g(s; t, t') = f(s; t^{-1}, t'^{-1})$.

COROLLARY 5.6. *If π is a conformal projectivity of the group A upon the group B, if the element t in A is strictly independent of the element s in A, and if $\{t'\} = \{t\}^{\pi}$, then there exists one and only solution s' of the equations* (5.5.1).

Proof. It is a consequence of Theorem 5.1 that $\{t\}$ is strictly independent of $\{s\}$. The elements s and t generate an abelian subgroup $\{s, t\}$ of A; and abelian groups are A-groups by Theorem 5.3. Hence we may apply Lemma 5.5 to the conformal projectivity π of $\{s, t\}$, proving the corollary.

COROLLARY 5.7. *If π is a conformal projectivity of the A-group A upon the group B, if the cyclic subgroup $\{t\}$ of A is strictly independent of the subgroup S of A, and if $\{t'\} = \{t\}^{\pi}$, then $g(s; t, t', \pi)$ defines a $1:1$ correspondence between the elements in S and those in S^{π}.*

Proof. If Z is a cyclic subgroup of S, then $\{t\}$ is strictly independent of Z; and it follows from the considerations in the proofs of Theorem 5.4 and Lemma 5.6 that $g(s; t, t', \pi)$ maps the $n^*(Z)$ elements in Z which generate Z upon the $n^*(Z^{\pi}) = n^*(Z)$ elements in Z^{π} which generate the cyclic group Z^{π}. Thus $g(S; t, t', \pi) = S^{\pi}$ and g effects a $1:1$ correspondence between the elements in S and those in S^{π}.

LEMMA 5.8. *If π is a conformal projectivity of the A-group A upon the group B, if the element t^{-1} in A domineers the subgroup $\{u, v\}$ of A, if $\{t'\} = \{t\}^{\pi}$, and if r is the uniquely determined element in $\{u, v\}$, satisfying: $g(r; t, t', \pi) = g(u; t, t', \pi)g(v; t, t', \pi)$, then $\{u\} = \{rv^{-1}\}$.*

Proof. We note first that each of the cyclic subgroups $\{t^{-1}\}$, $\{vt^{-1}\}$ and $\{rt^{-1}\}$ is strictly independent of $\{u, v\}$. Using the abbreviation $x' = g(x; t, t', \pi)$ for x in $\{u, v\}$, and remembering that, by Lemma 5.5, $g(rv^{-1}; tv^{-1}, t'v'^{-1}, \pi)$ is well determined, we infer from equations (5.5.1) that

$$\{g(rv^{-1}; tv^{-1}, t'v'^{-1}, \pi)(t'v'^{-1})^{-1}\} = \{rv^{-1}(tv^{-1})^{-1}\}^{\pi} = \{rt^{-1}\}^{\pi}$$
$$= \{r't'^{-1}\} = \{u'v't'^{-1}\} = \{u'(t'v'^{-1})^{-1}\}.$$

The quotient $g(rv^{-1}; tv^{-1}, t'v'^{-1}, \pi)u'^{-1}$ is therefore contained in the cross cut of $\{u, v\}^{\pi}$ and $\{rt^{-1}\}^{\pi}$. But this cross cut is 1, since $\{rt^{-1}\}$ is strictly independent of $\{u, v\}$, as has been pointed out before. It follows that $u' = g(rv^{-1}; tv^{-1}, t'v'^{-1}, \pi)$ and $\{u\} = \{rv^{-1}\}$ is an immediate consequence of (5.5.1).

THEOREM 5.9. *If π is a conformal projectivity of the A-group A upon the group B, if the element t^{-1} in A domineers the subgroup S of A, if the*

degree of S is not less than 2, and if $\{t'\} = \{t\}^\pi$, then $g(s; t, t', \pi)$ is a coset preserving crossed isomorphism of S upon the subgroup S^π of A which induces π in the system of subgroups of S.

This is an immediate consequence of Corollary 5.7, Lemma 5.8 and Corollary 4.6.

We derive a partial converse of the last theorem. We note that every coset preserving crossed isomorphism of the group A upon the group B induces a projectivity which is index preserving in the strict sense; and we shall use the same symbol for the crossed isomorphism and for the projectivity it induces.

THEOREM 5.10. *If ϕ is a coset preserving crossed isomorphism of the A-group A upon the group B, if the cyclic subgroup $\{t\}$ of A is strictly independent of the cyclic subgroup $\{s\}$ of A, then $s^\phi = g(s; t, t^\phi, \phi)$.*

Proof. We deduce from Corollary 4.6 that $\{st^{-1}\}^\phi = \{s^\phi(t^\phi)^{-1}\}$; and we infer that $\{s\}^\phi = \{s^\phi\}$ from the conformality of ϕ (see Theorem 3.1 and Lemma 3.1.1). Thus s^ϕ is a solution of the equations (5.5.1). But it is a consequence of Lemma 5.5 that $g(s; t, t^\phi, \phi)$ is the only solution of the equations (5.5.1). Hence $s^\phi = g(s; t, t^\phi, \phi)$.

The following property of the function $g(\cdot\cdot\cdot)$ will be needed in future applications:

LEMMA 5.11. *If π is a conformal projectivity of the A-group A upon the group B, if the element t^{-1} in A domineers the subgroup $\{u, v\}$ of A, if $\{v\}$ is strictly independent of $\{u\}$, and if $\{t'\} = \{t\}^\pi$, then*

$$g(u; v, g(v; t, t', \pi), \pi) = g(u; t, t', \pi).$$

Proof. Suppose that T is a cyclic subgroup of $\{u, t\} \cap \{uv^{-1}, vt^{-1}\}$ which meets the requirements (A'') both with regard to $\{t\}$, $\{u\}$ and with regard to $\{vt^{-1}\}$, $\{uv^{-1}\}$. (Note that $\{t\}$ is strictly independent of $\{u\}$ and $\{vt^{-1}\}$ is strictly independent of $\{uv^{-1}\}$, since t^{-1} domineers $\{u, v\}$.) Since A is an A-group, the properties (A') and (A'') are equivalent. Hence there exist integers i, j such that $\{u\} = \{u^i\}$, $\{uv^{-1}\} = \{(uv^{-1})^j\}$ and $T = \{u^i t^{-1}\} = \{(uv^{-1})^j vt^{-1}\}$. It is a consequence of (5.iii) that $T \cap \{u, v\} = 1$; and hence it follows that $u^i = (uv^{-1})^j v$ or $u^i v^{-1} = (uv^{-1})^j$. Consequently $\{u^i v^{-1}\} = \{uv^{-1}\}$. But $\{v\}$ is strictly independent of $\{u\}$ and $\{uv^{-1}\}$ meets the requirement (A') and therefore (A'') with regard to $\{v\}$, $\{u\}$. Thus $\{uv^{-1}\} \cap \{u\} = 1$ by (5.iii) and consequently $u = u^i$, proving that $T = \{ut^{-1}\}$. Since $ut^{-1} = uv^{-1}vt^{-1}$, the following statement is readily verified:

$\{ut^{-1}\}$ is the only cyclic subgroup of A which meets the requirement (A'') both with regard to $\{t\}$, $\{u\}$ and with regard to $\{vt^{-1}\}$, $\{uv^{-1}\}$.

It is a consequence of Theorem 5.4 that B is an A-group.

We put $x' = g(x; t, t', \pi)$ for x in $\{u, v\}$. Then $\{vt^{-1}\}^\pi = \{v't'^{-1}\}$ and $\{ut^{-1}\}^\pi = \{u't'^{-1}\}$. If we put $u'' = g(u; v, v', \pi)$, as we may, since $\{v\}$ is strictly independent of $\{u\}$, then $\{uv^{-1}\}^\pi = \{u''v'^{-1}\}$. Hence it follows from the result of the first paragraph of this proof that $\{u't'^{-1}\}$ is the only cyclic subgroup of B which meets the requirement (A'') with regard to both $\{t'\}$, $\{u'\} = \{u''\}$ and with regard to $\{v't'^{-1}\}$, $\{u''v'^{-1}\}$. But $u''t'^{-1} = u''v'^{-1}v't'^{-1}$ so that $\{u''t'^{-1}\}$ meets the requirement (A') and therefore the requirement (A'') with regard to these two pairs of cyclic subgroups of B. Hence $\{u't'^{-1}\} = \{u''t'^{-1}\}$; and $u' = u''$ is derived as usual from (5. iii).

6. Crossed isomorphisms inducing a given projectivity. The three elements r, s, t in the group A are termed *a domineering triplet of* A, if they satisfy the following conditions:

(6. i) *If u and v are elements in A, then at least one of the three elements r, s, t domineers $\{u, v\}$.*

(6. ii) *If x, y, z is a permutation of the three elements r, s, t, if both $\{x\}$ and $\{y\}$ are strictly independent of the cyclic subgroup U of A, if x does not domineer $U\{y\}$ and y does not domineer $U\{x\}$, then z domineers both $U\{x\}$ and $U\{y\}$.*

If x, y, z is a permutation of the three elements r, s, t, then it follows from (6. i) that x domineers $\{y, z\}$. If U is a cyclic subgroup of A, then it follows from (6. i) that at least one of the three elements r, s, t, say r, domineers U; and it follows from (6. i) that at least one of the three elements r, s, t domineers $U\{r\}$. But this latter element cannot be r, showing that at least two distinct ones of the elements r, s, t domineer the cyclic subgroup U.

The following criterion will give an indication of the frequency of groups possessing domineering triplets.

THEOREM 6. 1. *The elements r, s, t in the center $C(A)$ of the R-group A form a domineering triplet of A if, and only if, the following conditions are satisfied by them:*

(a) *$n(r) = n(s) = n(t)$ is either 0 or a power of a prime;*

(b) *the elements r, s, t form a basis of $\{r, s, t\}$;*

(c) *$n(x)$ is a divisor of $n(r)$ for every x in A;*

(d) *if u and v are elements in A, then $\{u, v\} \cap \{r, s, t\}$ is generated by two elements.*

Remark. If A happens to be an abelian group, then condition (d) may be omitted.

Proof. Suppose first that the three elements r, s, t in $C(A)$ form a domineering triplet of the R-group A. Since $\{r, s, t\} = T$ is abelian, as a subgroup of the center, it follows from Theorem 5.1 that t domineers $\{r, s\}$ if, and only if, t is strictly independent of $\{r, s\}$; and now it is readily seen that the elements r, s, t are of equal order n and form a basis of T. If $0 \neq n = n'n''$ where n' and n'' are relatively prime, and if x is one of the elements r, s, t, then $x = x'x''$ where $n' = n(x')$ and $n'' = n(x'')$. If $1 \neq n'$, then $\{r', s''t'\}$ is certainly not domineered by r or t, since it contains both r' and t' which are powers, different from 1, of r and t respectively. Hence s domineers $\{r', s''t'\}$; and since this group contains the power s'' of s, it follows that $s'' = 1$ and that therefore $n'' = 1$. This completes the proof of the necessity of (a) and (b). The necessity of (c) is an almost immediate consequence of (6. i) and Theorem 5. 1. If, finally, u and v are elements in A, then one of the three elements r, s, t, say r, domineers $\{u, v\}$. But then r is strictly independent of the cross cut $\{u, v\} \cap \{r, s, t\}$ and now (d) is a consequence of (a), (b), (c).

Suppose, conversely, that the conditions (a) to (d) are satisfied by the triplet r, s, t in $C(A)$. If u and v are elements in A, then we infer from (a), (b) and (d) the existence of one element, say r, in the triplet r, s, t such that r is strictly independent of $\{u, v\} \cap \{r, s, t\}$. Since r is in $C(A)$, it follows from (c) that r is strictly independent of every element in $\{u, v\}$; and we deduce from Theorem 5. 2 that r domineers $\{u, v\}$, since A is an R-group. Thus (6. i) is satisfied by r, s, t. If U is a cyclic subgroup of A, then TU is abelian, since $T = \{r, s, t\} \leq C(A)$. Thus $U\{r\}$ is domineered by s if, and only if, $U\{r\} \cap \{s\} = 1$. But $U\{r\} \cap \{s\} = 1$ if, and only if, $\{r, s\} \cap U = 1$. Finally $\{r, s\} \cap U \neq 1$ and $\{r, t\} \cap U \neq 1$ if, and only if, $U \cap \{r\} \neq 1$; and now it is readily verified that (6. ii) is satisfied by r, s, t, since it is impossible that both $U \cap \{r\} \neq 1$ and $U \cap \{s\} \neq 1$.

THEOREM 6. 2. *If A is an A-group of degree not less than 2, if r^{-1}, s^{-1}, t^{-1} is a domineering triplet of A, if π is a conformal projectivity of A upon the group B, and if $\{t'\} = \{t\}^{\pi}$, then there exists one and only one coset preserving crossed isomorphism of A upon B which maps t upon t' and which induces π in the system of subgroups of A.*

Remark. It is a consequence of this theorem that π is not only conformal, but index preserving in the strict sense.

Proof. If ϕ and γ are coset preserving crossed isomorphisms which map t upon t' and which induce π in the system of subgroups of A, then it follows from Theorem 4.4 that $\eta = \phi\gamma^{-1}$ is an integral automorphism of A which leaves t invariant. It is a consequence of Theorem 5.10 that $x^{\eta} = g(x; y, y^{\eta}, 1)$, if $\{y\}$ is strictly independent of $\{x\}$; and we deduce from Lemma 5.5 that $x = g(x; y, y, 1)$, if $\{y\}$ is strictly independent of $\{x\}$. From these facts we derive first that $r = r^{\eta}$ and $s = s^{\eta}$ and that therefore $x = x^{\eta}$ for every x in A. Consequently there exists at most one coset preserving crossed isomorphism, meeting our requirements.

To prove the existence of at least one coset preserving crossed isomorphism with the desired properties, we proceed as follows: since t^{-1} domineers $\{r, s\}$, it follows from Lemma 5.5 that $r' = g(r; t, t', \pi)$ and $s' = g(s; t, t', \pi)$ are well determined. Since $\{s\}$ is strictly independent of $\{r\}$, it follows from Lemma 5.11 that $r' = g(r; s, s', \pi)$; and likewise we see that $s' = g(s; r, r', \pi)$. From

$$\{g(t; r, r', \pi)r'^{-1}\} = \{tr^{-1}\}^{\pi} = \{rt^{-1}\}^{\pi} = \{r't'^{-1}\} = \{t'r'^{-1}\}$$

and $\{t'\} = \{t\}^{\pi} = \{g(t; r, r', \pi)\}$ we deduce, as usual, that $t' = g(t; r, r', \pi)$ and likewise we show that $t' = g(t; s, s', \pi)$. Thus we have shown:

if x and y are two distinct elements of the three elements r, s, t, then $x' = g(x; y, y', \pi)$.

If x is any element in A, then at least one of the three elements r, s, t generates a cyclic subgroup strictly independent of $\{x\}$. If both $g(x; r, r', \pi)$ and $g(x; s, s', \pi)$ are defined, then we deduce from (6.ii) and Lemma 5.11 that $g(x; r, r', \pi) = g(x; s, s', \pi)$; and thus we have shown:

if u is any element in A, then of the three functions: $g(x; r, r', \pi)$, $g(x; s, s', \pi)$ and $g(x; t, t', \pi)$ at least one is defined for $x = u$; and all those that are defined have the same value u^{ϕ}.

If u and v are two elements in A, then there exists one element among the three elements r, s, t, say r, such that r^{-1} domineers $\{u, v\}$. Then $u^{\phi} = g(u; r, r', \pi)$ and $v^{\phi} = g(v; r, r', \pi)$. We deduce from Corollary 5.7 that $u = v$ if, and only if, $u^{\phi} = v^{\phi}$ and that ϕ effects a $1:1$ correspondence of $\{u, v\}$ upon $\{u, v\}^{\pi}$; and we infer from Lemma 5.8 that $\{uv^{-1}\} = \{u^{\phi}, (v^{\phi})^{-1}\}$. Now we readily deduce from Corollary 4.6 that ϕ is a coset preserving crossed isomorphism of A upon B which maps t upon t' and which induces π in the system of subgroups of A.

THEOREM 6. 3. *The group A possessing a domineering triplet is abelian if, and only if, A is an A-group containing with any three elements an element strictly independent of each of them.*[20]

Proof. If the abelian group A possesses a domineering triplet, then it follows from Theorem 5. 3 that A is an A-group; and it is readily inferred from Theorem 6. 1 that A contains either three independent elements of order 0 or three independent elements of maximum order p^m ($A^{p^m} = 1$). From this latter fact we easily deduce that there exists to any three given elements x, y, z an element w which is strictly independent of x and y and z.

Suppose now, conversely, that A is an A-group containing with any three elements an element strictly independent of each of them; and let r, s, t be a domineering triplet of A. There exists an element u strictly independent of r and s and t. Since r cannot domineer $\{r, u\}$, it follows from (6. i) that either s or t domineers $\{r, u\}$. If s domineers $\{r, u\}$, then we deduce from $u^s = su$ and Theorem 5. 1 that s is strictly independent of u and that therefore $n(s)$ is a multiple of $n(u)$. Since $n(u)$ is a multiple of $n(r)$ and $n(t)$, we have shown that $n(s)$ is a multiple of $n(r)$ and $n(t)$. This proves the existence of one element among the three elements r, s, t whose order is a multiple of the orders of the other two elements.

Without loss in generality we assume now that $n(t)$ is a multiple of both $n(r)$ and $n(s)$. If x is an element in A, then there exists an element z in A which is strictly independent of x and of r and of s. Since t cannot domineer $\{t, z\}$, one of the two elements r and s domineers $\{t, z\}$. If e. g., r domineers $\{t, z\}$, then we deduce from $rz = zr$ and Theorem 5. 1 that r is strictly independent of z. Hence $n(r)$ is a multiple of $n(z)$. Since $n(t)$ is a multiple of $n(r)$, and since $n(z)$ is a multiple of $n(x)$, we have shown that

$$n(t) \text{ is a multiple of } n(x) \text{ for every } x \text{ in } A.$$

Since A is an A-group of degree not less than 2, we deduce from Theorem 6. 2 the existence of a coset preserving crossed isomorphism ϕ such that $t^\phi = t^{-1}$ and $S^\phi = S$ for every subgroup S of A. Since the coset preserving crossed isomorphism $\gamma = 1$ satisfies $S^\gamma = S$ too, we deduce from Theorem 4. 3 that ϕ is an integral automorphism of A. If x is any element in A, then there exists an element y in A which is strictly independent of both t and x. Clearly $n(t) = n(y)$. Since t and y generate an abelian subgroup, it follows from Theorem 1. 5 that $y^\phi = y^{-1}$. Since y and x generate an abelian subgroup, and

[20] This property is not much stronger than the requirement of being of degree not less than 2.

since y is strictly independent of x, it follows from Theorem 1. 5 that $x^{\phi} = x^{-1}$. Thus we have shown that mapping every element in A upon its inverse constitutes an automorphism of A; and this property is a characteristic property of abelian groups. Hence A is abelian.

COROLLARY 6. 4. *If A is an A-group containing a domineering triplet, if there exists to any three elements x, y, z in A an element w which is strictly independent of x and of y and of z, and if π is a conformal projectivity of A upon the group B, then the following two properties of π and B imply each other:*

(a) *π is induced by an ordinary isomorphism of A upon B.*

(b) *If B does not contain elements of order* 0, *but contains elements of order n, then the center of B contains elements of order n.*

Proof. It is a consequence of Theorems 6. 2 and 6. 3 that A is an abelian group and that π is induced by a coset preserving crossed isomorphism of A upon B. If (a) holds true, then B is abelian too and condition (b) is certainly satisfied by B. If, conversely, (b) is satisfied by B, and if π is induced by the coset preserving crossed isomorphism ϕ of A upon B, then ϕ is an ordinary isomorphism of A upon B, as follows from Theorem 4. 1, and condition (b).

Remark. It may be shown that the cyclic subgroups, generated by the elements in a domineering triplet of an A-group, are mapped by conformal projectivities upon cyclic subgroups, generated by the elements in a domineering triplet, since conformal projectivities map A-groups upon A-groups. On the other hand it will be shown in the next section that A-groups containing domineering triplets and containing with any three elements an element strictly independent of these may be mapped by coset preserving crossed isomorphisms and therefore by conformal projectivities upon not abelian groups, in spite of the fact that they are, by Theorem 6. 3, abelian. Thus of all the concepts discussed only one is not invariant under conformal projectivities, namely the concept of an element strictly independent of an element (or a set of elements).

7. Coset preserving crossed isomorphisms and conformal projectivities of abelian groups. If A is an abelian group, ϕ a coset preserving crossed isomorphism of A upon the group B, and if A contains elements of order 0, then A is, as an abelian group, generated by the elements of order 0 in A; and it follows from Theorem 3. 3 that ϕ is an ordinary isomorphism.

If A is an abelian group without elements of order 0, then A is known to be the direct product of its primary components A_p where A_p is the subgroup of all the elements of order a power of the prime number p. If ϕ is a coset preserving crossed isomorphism of A upon the group B, then $B_p = A_p{}^{\phi}$ is a subgroup of B and consists of all the elements of order a power of p, since ϕ is conformal and since A_p consists just of the elements of order a power of p in A. Thus each B_p is a normal subgroup of B. Since $\prod_{q \neq p} B_q = \prod_{q \neq p} A_q{}^{\phi}$ consists of the elements of order prime to p and is a subgroup of B, it follows that B is the direct product of its primary components B_p.

The preceding considerations show that for a study of coset preserving crossed isomorphisms of abelian groups it suffices to consider the coset preserving crossed isomorphisms of abelian p-groups.

If G is a p-group the orders of whose elements are bounded, then we denote by $p^{m(G)}$ *the maximum order of the elements in* G.

THEOREM 7.1. *The p-group G may be obtained from an abelian p-group by a coset preserving crossed isomorphism which is not an ordinary isomorphism if, and only if, the following conditions are satisfied by* G:

(1) *The orders of the elements in* G *are bounded.*

(2) *There exists a homomorphism* $e(x)$ *of* G *into the group of integers modulo* $p^{m(G)}$, *prime to* p, *meeting the following requirements*:

(2. i) *The normal subgroup* E *of all the elements* x *in* G *such that* $e(x) \equiv 1$ modulo $p^{m(G)}$ *is abelian and is different from* G.

(2. ii) *If* x *is in* G *and* y *is in* E, *then* $x^{-1}yx = y^{e(x)}$.

(2. iii) *If* $p = 2$, *then* $2 < m(G)$ *and* $e(x) \equiv 1$ *modulo* 4 *for every* x *in* G.

Remarks. 1. If p is odd, then it follows from Theorem 3.1 and Theorem 3.3, (b) that every integral crossed isomorphism is coset preserving. 2. It is a consequence of this theorem that the groups, recently discussed by Beaumont (2), may be obtained from abelian p-groups by coset preserving crossed isomorphisms.

Proof. A. If there exists an abelian p-group A and a coset preserving crossed isomorphism ϕ of A upon G, then there exists to every x in A an integral automorphism $e(x)$ of A such that ϕ is an $e(x)$-isomorphism of A upon G. If ϕ is not an ordinary isomorphism, then $e(x) \neq 1$ for some x in A.

The function $e'(x) = e(x^{\phi^{-1}})$ for x in G is by Corollary 1.2, (iii) a

homomorphism of G into the group of integral automorphisms of A. The set E of all the elements x in A, satisfying $e(x) = 1$, is a subgroup of A and E^ϕ is the kernel of the homomorphism $e'(x)$. Hence $E' = E^\phi$ is a normal subgroup of G. Clearly ϕ is an ordinary isomorphism of E upon E' so that E' is abelian. Since ϕ is assumed not to be an ordinary isomorphism, we have $E < A$ and therefore $E' < G$.

If the orders of the elements in G are not bounded, then neither are the orders of the elements in A. In this case we deduce from Theorem 1. 5 that the group I^*_A of the integral automorphisms of A is essentially the same as the group of p-adic integers, prime to p. Since $e'(x)$ is a homomorphism of G into I^*_A, and since every element in G is of order a power of p, every $e'(x)$ is essentially an integral p-adic number, prime to p, of multiplicative order a power of p. But there do not exist integral p-adic numbers of multiplicative order a power of p apart from ± 1; and -1 is of order 2. This shows already the necessity of condition (1) for odd p. If $p = 2$, then $e(x) = \pm 1$, as has been remarked already. We infer from Theorem 2. 1 that E is of index 2 in A; and it follows from Corollary 2. 4 that ϕ maps the elements not in E upon elements of order 2. The elements in A, but not in E, are therefore of order 2 too, since ϕ is conformal. But A is abelian and hence it would follow that $A^2 = 1$, an obvious contradiction. Thus we have completed the proof of the necessity of (1).

From (1) and the facts already proven we infer that $e'(x)$ is a homomorphism of G into the group of integers modulo $p^{m(A)}$, prime to p, which satisfies condition (2. i), since $m(A) = m(G)$ is a consequence of the conformality of ϕ. The necessity of (2. ii) is a consequence of Corollary 1. 7; and the necessity of (2. iii) may be deduced from Theorem 3. 4, (1) (together with $E' < G$).

B. Suppose, conversely, that conditions (1), (2) are satisfied by the p-group G. We denote by $e(x)$ some homomorphism of G meeting the requirement (2) and by E the kernel of $e(x)$. Then we infer from (2. iii) and from well known facts [21] that G/E is a cyclic group of order p^n where $0 < n < m(G)$; and in case $p = 2$ we have $n < m(G) - 1$. Thus there exists an element q in G which generates G modulo E; $r = q^{p^n}$ is an element in E; $e = e(q)$ is an integer modulo $p^{m(G)}$, prime to p, whose multiplicative order is exactly p^n; and thus we deduce from (2. i) and (2. ii) the following fact:

(7. 1. 1) *The group G is generated by adjoining to the abelian p-group E an element q, subject to the following relations*:

[21] See e. g. Hecke (1), § 13.

$$q^{p^n} = r, \quad q^{-1}xq = x^e \text{ for } x \text{ in } E.$$

Since e is of multiplicative order p^n modulo $p^{m(G)}$, and since it follows from (2. iii) that $e \equiv 1$ modulo 4, if $p = 2$, we may represent e in the form: $e \equiv 1 + e_0 p^{m(G)-n}$ modulo $p^{m(G)}$ where e_0 is prime to p. Hence

$$(1 + e + \cdots + e^{p^n-1})p^{-n} \equiv (e^{p^n} - 1)(e-1)^{-1}p^{-n} \equiv e^* \text{ modulo } p^{m(G)}$$

where e^* is prime to p. Consequently there exists one and only one element r^* in the abelian p-group E such that $r = r^{*e^*}$; and there exists one, and essentially only one, abelian p-group A which is obtained by adjoining to E an element d, subject to the relation: $d^{p^n} = r^*$. It is readily seen that $p^{m(G)}$ is the maximum order of the elements in A. Concerning A we need the following existence lemma.

(7. 1. 2) *There exists a single valued A to I^*_A function $f(x)$ such that $f(y) = 1$ for y in E, $x^{f(d)} = x^e$ for x in A and such that*

$$f(u)f(v) = f(u^{f(v)}v) \text{ for } u, v \text{ in } A.$$

To prove this lemma we remark first that every element in A may be represented in one and only one way in the form:

$$yd^i \text{ for } y \text{ in } E \text{ and } 0 < i \leq p^i; \text{ and we put } i = i(yd^i).$$

(Note that the representation of 1 is $r^{*-1}d^{p^n}$ with $i(1) = n$.) Since p^n is the multiplicative order of e modulo $p^{m(G)}$, it follows that $0 < j \leq p^n$ and $1 + e + \cdots + e^{j-1} \equiv 0$ modulo p^n imply $j = p^n$; and from this fact we deduce:

(7. 1. 3) If $0 < j, j' \leq p^n$, then $j = j'$ is a necessary and sufficient condition for $1 + e + \cdots + e^{j-1} \equiv 1 + e + \cdots + e^{j'-1}$ modulo p^n.

Consequently there exists to every element x in A one and only one integer $j(x)$ such that $0 < j(x) \leq p^n$ and $i(x) \equiv 1 + e + \cdots + e^{j(x)-1}$ modulo p^n; and thus a single valued A to I^*_A function $f(x)$ is defined by the equation:

$$y^{f(x)} = y^{e^{j(x)}} \text{ for every } y \text{ in } A.$$

Then we have essentially the identity $f(x) = e^{j(x)}$; and it is readily seen that $y^{f(d)} = y^e$ for y in A, and that $f(y) = 1$ for y in E. Furthermore we have $f(x) = f(d^{i(x)}) = f(d^{i(x)+kp^n})$ for every x in A and for every integer k. Consequently we find for elements u, v in A:

$$\begin{aligned} f(u^{f(v)}v) &= f(d^{i(u)f(v)+i(v)}) \\ &= f(d^{(1+e+\cdots+e^{f(u)-1})e^{f(v)}+1+e+\cdots+e^{f(v)-1}}) \\ &= f(d^{1+e+\cdots+e^{f(u)+f(v)-1}}) = e^{f(u)+f(v)} = e^{f(u)}e^{f(v)} \\ &= f(d^{i(u)})f(d^{i(v)}) = f(u)f(v)\,; \end{aligned}$$

and thus we have shown that the function $f(x)$ meets all the requirements of (7.1.2).

If $f(x)$ is some function meeting the requirements of (7.1.2), then there exists, by Theorem 1.1, an $f(x)$-isomorphism γ of A upon some group $B = A^{\gamma}$. It is a consequence of Theorem 1.5 that γ is an integral crossed isomorphism; and we deduce from Theorem 3.1, condition (2.iii) and Theorem 3.4, that γ is coset preserving. Hence B is a p-group, satisfying $m(B) = m(A) = m(G)$. It follows from Corollary 1.7 that $(d^{\gamma})^{-1}y^{\gamma}d^{\gamma} = (y^{\gamma})^{e}$ for y in E; and we note that E consists of all the elements y in A such that $f(y) = 1$ and that therefore γ effects an isomorphism of E upon E^{γ}, a subgroup of B. Furthermore we find that

$$(d^{\gamma})^{p^n} = (d^{1+c+\cdots+c^{p^n-1}})^{\gamma} = (r^{*c^*})^{\gamma} = r^{\gamma}.$$

Hence there exists by (7.1.1) an isomorphism κ of B upon G such that $x^{\gamma\kappa} = x$ for x in E and $d^{\gamma\kappa} = q$; and it is readily verified that $\gamma\kappa$ is a coset preserving crossed isomorphism of A upon G which is not an ordinary isomorphism, as was to be shown.

THEOREM 7.2. *The non-abelian p-group G may be obtained from an abelian p-group by a coset preserving crossed isomorphism if, and only if, there exists a normal abelian subgroup E of G with the following properties*:

(i) *G/E is a cyclic group.*

(ii) *Every subgroup of E is a normal subgroup of G.*

(iii) *If b is an element of order 4 in E, then b is in the center of G.*

Proof. A. If there exists an abelian p-group A and a coset preserving crossed isomorphism ϕ of A upon G, then ϕ is not an ordinary isomorphism, since A is abelian and G is not. Hence it follows from Theorem 7.1 that the orders of the elements in G are bounded; and that, if we denote by p^m the common maximum order of the elements in A and in G, there exists a homomorphism $e(x)$ of G into the group of integers modulo p^m, prime to p, with the following properties:

(a) the kernel E of the homomorphism $e(x)$ is a normal abelian subgroup of G;

(b) if x is in G and y is in E, then $x^{-1}yx = y^{e(x)}$;

(c) if $p = 2$, then $2 < m$ and $e(x) \equiv 1$ modulo 4 for every x in G.

From (a) and (c) we infer that (i) is satisfied by E; and (b) implies property (ii). If, finally, b is an element of order 4 in E, then $p = 2$ and we deduce from (b) and (c) that $x^{-1}bx = b$ for every x in G, since $p = 2$ implies $e(x) \equiv 1$ modulo 4. Thus we have verified the necessity of our conditions.

B. Suppose, conversely, that there exists an abelian normal subgroup E of the non abelian p-group G, meeting the requirements (i) to (iii). Then there exists only a finite number of subgroups between E and G; and thus we may assume without loss in generality that the following (maximum-) condition is satisfied by E and G:

(iv) *If the abelian subgroup V between E and G meets the requirements* (i) *to* (iii), *then $E = V$.*

It is a consequence of (ii) that transformation by an element in G induces an integral automorphism in E. If the orders of the elements in G were not bounded, then the orders of the elements in E would not be bounded. In this case we could deduce from the commutativity of E and from Theorem 1.5 the existence of an integral p-adic number $e(x)$, prime to p, such that $x^{-1}yx = y^{e(x)}$ for x in G and y in E. The function $e(x)$ constitutes a homomorphism of G into the group of p-adic numbers, prime to p. Since x is of order a power of p, it follows that the multiplicative order of $e(x)$ is a power of p too. This implies $e(x) = 1$ for odd p, and $e(x) = \pm 1$ for $p = 2$. But $p = 2$, $e(x) = -1$ are excluded by (iii), since E contains elements of order 4. Consequently E is part of the center of G. From (i) we may deduce now that G is abelian, contradicting our hypothesis. Thus we have shown:

The orders of the elements in G are bounded.

(This is precisely condition (1) of Theorem 7.1.) We denote by p^m the maximum order of the elements in G, by $p^{m'}$ the maximum order of the elements in E and by p^n the order of the cyclic group G/E.

Let V be a coset of G modulo E, generating G/E. Since E is abelian it follows from (ii) that every element in the coset V induces (by transformation) the same integral automorphism of E. Hence we infer from Theorem 1.5 the existence of an integer e, prime to p, such that $v^{-1}xv = x^e$ for every x in E and for every v in V. Since G is not abelian, we have $0 < m'$ and $e \not\equiv 1$ modulo $p^{m'}$. Since V is of order p^n, it follows from well known

facts[22] that $e = 1 + e'p^j$ where e' is prime to p and where $0 < j < m' \leq m$. In case $p = 2$ we deduce from the non-commutativity of G and from (iii) the additional restriction: $1 < j$ and $2 < m'$.

There exists in G an element z of order p^m. If v is an element in the coset V, then $z = uv^i$ for u in E, since $Ez = V^i$ for suitable i. Thus

$$1 \neq z^{p^{m-1}} = (uv^i)^{p^{m-1}} = u^r v^{ip^{m-1}} \text{ for } r \equiv 1 + e^{-i} + \cdots + e^{-i(p^{m-1}-1)} \text{ modulo } p^m$$

and it is readily seen that $r \equiv 0$ modulo p^{m-1}. Thus at least one of the elements u and v is of order p^m. If $v^{p^{m-1}} = 1$, then $(uv)^{p^{m-1}} = u^s$ for $s \equiv 1 + e^{-1} + \cdots + e^{-(p^{m-1}-1)}$ modulo p^m; and one verifies that $s \not\equiv 0$ modulo p^{m-1}. Thus uv is of order p^m, if v is not; and we have shown that V contains an element g of order p^m. Since g^{p^n} belongs to E, it follows that $g^{p^n} = g^{-1}g^{p^n}g = g^{p^n e}$ and that therefore $p^n \equiv p^n e \equiv p^n + e'p^{j+n}$ modulo p^m. Since e' is prime to p, this implies $m \leq n + j$ or $m - n \leq j$.

If $m - n < j$, then we would deduce from $e^{p^{n-1}} \equiv 1 + e''p^{j+n-1}$ modulo p^m that $e^{p^{n-1}} \equiv 1$ modulo p^m, and that $p^{n-1}e \equiv p^{n-1} + e'p^{j+n-1} \equiv p^{n-1}$ modulo p^m. The subgroup W of G which is generated by adjoining $g^{p^{n-1}}$ to E is therefore an abelian normal subgroup of G such that G/W is cyclic of order p^{n-1} and such that $g^{-1}xg = x^e$ for x in W. Thus we readily verify that W meets all the requirements (i) to (iii) in spite of $E < W$, contradicting (iv). Hence $m - n \nless j$, proving that $j = m - n$.

Thus the integer e, prime to p, has multiplicative order p^n modulo p^m. If we put $e(x) \equiv e^i$ modulo p^m, if $Ex = V^i$, then this function $e(x)$ clearly constitutes an isomorphism of G/E into the group of integers modulo p^m, prime to p. This homomorphism $e(x)$ of G has the kernel E and meets the requirement (2) of Theorem 7.1 by (iii) and the choice of e. Hence it follows from Theorem 7.1 that there exists an abelian p-group A and a coset preserving crossed isomorphism of A upon G, as we desired to prove.

We note that we have proved the following inequalities:

$$0 < m - n < m' \leq m, \text{ if } p \text{ is odd; and } 1 < m - n < m' \leq m \text{ for } p = 2.$$

To apply the theorems of the preceding section to abelian groups it will be convenient to introduce the following concept: *the group A is of degree not less than* 3, if there exists to every pair of elements u, v in A an element w in A which is strictly independent of the subgroup $\{u, v\}$. An abelian group A may be shown to be of degree not less that 3 if, and only if, it contains either three independent elements of order 0, or in case it does not contain

[22] See e. g. Hecke (1), § 13.

elements of order 0, but contains elements of order n, then it contains three independent elements of order n. The following criterion follows now from Theorem 6. 1:

The abelian group A contains a domineering triplet if, and only if, A is of degree not less than 3 and contains elements of order 0 or is a p-group the orders of whose elements are bounded. If such an abelian group A contains elements of order 0, then every element of order 0 belongs to some domineering triplet; and if it is a p-group, then every element of maximum order $p^{m(A)}$ belongs to some domineering triplet. From these facts and Theorems 6. 2, 5. 3 and Corollary 6. 4 we deduce the following theorem:

THEOREM 7. 3. *If A is an abelian group of degree not less than 3, if A contains elements of order 0 or is a p-group the orders of whose elements are bounded, if t is an element in A such that $n(t)$ is a multiple of $n(x)$ for every element x in A, and if π is a conformal projectivity of A upon the group B and t' an element in B such that $\{t\}^\pi = \{t'\}$, then there exists one and only one coset preserving crossed isomorphism ϕ of A upon B which maps t upon t' and induces π in the system of subgroups of A. The crossed isomorphism*[23] *ϕ is an ordinary isomorphism of A upon B if, and only if, either A contains elements of order 0 or the center of B contains elements of maximum order in B.*

If A is an abelian group without elements of order 0, and if π is a conformal projectivity of A upon the group B, then π maps the primary components A_p of A upon the primary components B_p of B; and B is the direct product of its primary components B_p. Thus it suffices, on account of Theorem 7. 3, to consider the conformal projectivities of abelian p-groups the orders of whose elements are not bounded.

THEOREM 7. 4. *If the abelian p-group A is of degree not less than 3, and if the orders of the elements in A are not bounded, then every conformal projectivity of A upon some group B is induced by an ordinary isomorphism of A upon B.*

Proof. If $0 < k$, then we denote by $A(k)$ the subgroup of all the elements in A whose orders are divisors of p^k. Every $A(k)$ contains elements of order p^k and is of degree not less than 3. If π is a conformal projectivity of A upon the group B, then the subgroup $B(k) = A(k)^\pi$ consists of all the elements in B whose order is a divisor of p^k. The existence of crossed iso-

[23] For the validity of this last statement it suffices to assume that A be an abelian group of degree not less than 2.

morphisms of $A(k)$ upon $B(k)$ which induce π in the system of subgroups of $A(k)$ is an immediate consequence of Theorem 7.3. The main step of our proof will be the proof of the following lemma:

(7.4.1) *Every coset preserving crossed isomorphism of $A(k)$ upon $B(k)$ which induces π in the system of subgroups of $A(k)$ is induced by a coset preserving crossed isomorphism of $A(k+1)$ upon $B(k+1)$ which induces π in the system of subgroups of $A(k+1)$.*

From the existence of a coset preserving crossed isomorphism of $A(k+1)$ upon $B(k+1)$ and from Theorem 7.1 we infer the existence of an abelian normal subgroup N of $B(k+1)$ such that $B(k+1)/N$ is cyclic. If $M = N^{\pi^{-1}}$, then $A(k+1)/M$ is cyclic too. Consequently the degree of M is at least 2 and the maximum order of the elements in M is p^{k+1}. From the commutativity of M and N and from Theorem 4.1 we deduce that every coset preserving crossed isomorphism of M upon N is an ordinary isomorphism.

Let ϕ be a coset preserving crossed isomorphism of $A(k)$ upon $B(k)$ which induces π in the system of subgroups of $A(k)$. It has been remarked before that M contains an element t of order p^{k+1}. Then t^p is an element of order p^k in M and in $A(k)$; and $(t^p)^\phi$ is an element of order p^k in the subgroup $\{t\}^\pi$ of order p^{k+1} of N. Hence there exists an element t' of order p^{k+1} in N such that $\{t\}^\pi = \{t'\}$ and $t'^p = (t^p)^\phi$. There exists, by Theorem 7.3, one and only one coset preserving crossed isomorphism γ of $A(k+1)$ upon $B(k+1)$ which induces π in the system of subgroups of $A(k+1)$ and which maps t upon t'. Since, as has been pointed out before, γ effects an ordinary isomorphism of N upon M, we have in particular:

$$(t^p)^\gamma = (t^\gamma)^p = t'^p = (t^p)^\phi.$$

Thus γ and ϕ both induce π in the system of subgroups of $A(k)$ and they map the element t^p of maximum order in $A(k)$ upon the same element in $B(k)$. Hence it follows from Theorem 7.3 that γ and ϕ coincide on $A(k)$; and this completes the proof of (7.4.1).

From (7.4.1) and the fact that every element in A is in some $A(k)$ one readily deduces the existence of a 1:1 correspondence η mapping the elements in A upon the elements in B with the following properties:

η induces in every $A(k)$ a coset preserving crossed isomorphism.
η induces π in the system of subgroups of $A(k)$.

Since every finite subset of A is contained in some $A(k)$, we deduce from these properties and from Corollary 4.6 that

η induces π in the system of subgroups of A, and that

$$\{uv^{-1}\}^{\eta} = \{u^{\eta}(v^{\eta})^{-1}\} \text{ for } u, v \text{ in } A.$$

But then it follows from Corollary 4.6, applied in the opposite direction, that η is a coset preserving crossed isomorphism of A upon B which induces π in the system of subgroups of A. Since the orders of the elements in A are not bounded, it follows from Theorem 7.1 that η is an ordinary isomorphism of A upon B, inducing π, as was to be shown.

It is an immediate consequence of Theorems 7.1 and 7.2 that there exist coset preserving crossed isomorphisms of abelian p-groups of degree not less than 3 upon not abelian groups. Hence there exist projectivities which are index preserving in the strict sense of abelian groups of degree not less than 3 upon not abelian groups; and it is clearly impossible to induce these by ordinary isomorphisms. This patently contradicts Baer (1), Theorem 11.8, (a, 2). But it follows from Theorems 7.3 and 7.4 that in order to correct this statement it suffices to add in (a, 2) the words: "*provided the center of H contains an element of order* $p^{m(G)}$." The mistake leading to this oversight has been made in the proof and statement of Baer (1), (9.4) where commutativity of v and w is needed, though no such hypothesis has been made. The applications made of these theorems in Baer (1), **12** and **13** are not affected by these corrections, since in the proofs of Baer (1), Lemma 12.1 and Baer (1), Theorem 13.1 the needed hypotheses of commutativity are proven independently of the corrected results, and since we may apply in their stead, where necessary, the present Theorems 3.3, 3.4, 4.1, 5.9, 7.3, 7.4. On the other hand it should be pointed out that Baer (1), Theorem 13.1 is a better result than the present Theorem 7.3, as far as groups containing elements of order 0 are concerned. This part of the present Theorem 7.3 has only been stated, since its omission would not have changed or simplified any of the proofs that were needed anyway for the proof of independent results.

8. Structure of groups, obtained from abelian p-groups by integral crossed isomorphisms. If A is an abelian p-group, and if $p \neq 2$, then it follows from Corollary 3.6 that every integral crossed isomorphism of A upon a group B is conformal; and the maps of abelian p-groups under coset preserving crossed isomorphisms have been determined in **7**. Thus it suffices to consider the integral crossed isomorphisms of abelian 2-groups; and that these need not be conformal, has been mentioned in **2**.

If ϕ is an integral crossed isomorphism of the abelian 2-group A upon the group B, then we deduce from Theorem 3.5 that B is a 2-group. There

exists to every x in A an integral automorphism $e(x)$ of A such that ϕ is an $e(x)$-isomorphism of A upon B. It is a consequence of Theorem 1. 5 and Theorem 3. 4, (1) that ϕ is not conformal if, and only if, there exists an element w in A whose order is a multiple of 4 such that $x^{e(w)} = x^{-1+4i}$ for every x in A. If the orders of the elements in A are not bounded, then the group I^*_A of the integral automorphisms of A is essentially the same as the group of odd integral 2-adic numbers. It is a consequence of Corollary 1. 3 and the fact that B is a 2-group, that every $e(x)$ is of order a power of 2. But the only odd integral 2-adic numbers of order a power of 2 are ± 1; and the $e(x)$-isomorphisms with $e(x) = \pm 1$ have been completely determined in **2**. Thus we may assume without loss in generality that the orders of the elements in A are bounded. If we denote by 2^m the maximum order of the elements in A, then I^*_A is the group of odd integers modulo 2^m; and the above condition for non-conformality of ϕ asserts the existence of an element w such that $e(w) \equiv -1$ modulo 4. We denote by $A^* = A^*(\phi)$ the set of all the elements x in A such that $e(x) \equiv 1$ modulo 4 and by $A^{**} = A^{**}(\phi)$ the set of all the elements x in A satisfying: $e(x) = 1$.

LEMMA 8. 1. *If 2^m is the maximum order of the elements in the abelian 2-group A, if $e(x)$ is, for every x in A, an integral automorphism of A, if the $e(x)$-isomorphism ϕ of A upon the group B is not conformal, then $1 < m$, $A^*(\phi)$ and $A^{**}(\phi)$ are subgroups of A, A^* is of index 2 in A, A^*/A^{**} is a cyclic group, and ϕ is conformal on A^* and an ordinary isomorphism on A^{**}.*

Proof. If both u and v are elements in A^{**}, then it follows from Theorem 1. 1 that $e(uv) = e(u^{e(v)}v) = e(u)e(v) = 1$, showing that A^{**} is a subgroup of A. If u is an element in A^*, then we infer from (7. 1. 3) for every positive i the existence of a positive integer $j = j(i)$ such that $u^i = u^{1+e(u)+\cdots+e(u)^{j-1}}$. Hence it follows from Theorem 1. 1 that $e(u^i) = e(u^{1+e(u)+\cdots+e(u)^{j-1}}) = e(u)^j$. Thus A^* contains, with u, every power of u. If u and v are elements in A^*, then $u^{e(v)^{-1}}$ belongs to A^*; and we infer from Theorem 1. 1 that $e(uv) \equiv e(u^{e(v)^{-1}})e(v) \equiv 1$ modulo 4, showing that A^* is a subgroup of A.

It is a consequence of Theorem 1. 3 that $A^{*\phi}$ and $A^{**\phi}$ are subgroups of B; and we deduce from Theorem 3. 4 that ϕ is conformal on A^*. It is a consequence of Corollary 1. 3 that the homomorphism $d(x) = e(x^{\phi^{-1}})$ of B into I^*_A maps exactly $B^{**} = A^{**\phi}$ upon 1; and it maps exactly the elements in $B^* = A^{*\phi}$ upon elements of the subgroup of I^*_A which consists of integers $\equiv 1$ modulo 4. Since this latter group is a cyclic subgroup of index 2 in I^*_A, our contentions are now readily verified, since the non-conformality of ϕ implies $A^* < A$.

LEMMA 8. 2. *Suppose that the $e(x)$-isomorphism ϕ of the abelian 2-group A is not conformal, that the orders of the elements in A are bounded, and that 2^m is the maximum order of the elements in A $(1 < m)$.*

(a) *If B/B^{**} is cyclic (where $B = A^\phi$, $B^{**} = A^{**\phi}$, $B^* = A^{*\phi}$), then $A^* = A^{**}$ (and B/B^{**} and A/A^{**} are of order 2).*

(b) *If B/B^{**} is not cyclic, then either B/B^{**} and A/A^{**} are both of order 4 or else every element c in A such that $e(c) \equiv -1$ modulo 2^m meets the following requirements:*

(b′) *A is generated modulo A^{**} by c; and A^* is generated modulo A^{**} by c^2; and A/A^{**} is of order 2^{m-1};*

(b″) *$e(c^{2i}) \equiv 1 - 4i$ modulo 2^{m-1}.*

Remark. If B/B^{**} is not cyclic, then $2 < m$ and there exist elements c in A such that $e(c) \equiv -1$ modulo 2^m, since B/B^{**} is mapped isomorphically by $e(x)$ upon a group of odd integers modulo 2^m.

Proof. A. If B/B^{**} is cyclic, then there exists an element d in A such that B is generated modulo B^{**} by $b = d^\phi$. Since A^* is a group between A^{**} and A, and since A^* is of index 2 in A, it follows that d is not in A^*, that d^2 is in A^* and that $e(d) \equiv -1$ modulo 4. Since B/B^{**} is cyclic, since B^* is a subgroup between B^{**} and B such that B/B^* is of order 2, and since B is generated modulo B^{**} by b, it follows that B^* is generated modulo B^{**} by $b^2 = (d^{1+e(d)})^\phi$. Denote by V the subgroup of A which is obtained by adjoining $d^{1+e(d)}$ to A^{**}. Clearly V^ϕ contains B^{**} and b^2 and therefore B^*. Hence $A^* \leq V$ so that, in particular, d^2 is in V. Since $1 + e(d) \equiv 0$ modulo 4, there exists, consequently, an integer k and an element v in A^{**} such that $d^2 = d^{4k}v$ or $d^{(1-2k)2} = v$. But $1 - 2k$ is odd and hence d^2 is in A^{**}. But this shows that $d^{1+e(d)}$ is in A^{**} and that therefore b^2 is in B^{**}. Since B^* is obtained by adjoining b^2 to B^{**}, we deduce now that $B^* = B^{**}$ and $A^* = A^{**}$.

B. If B/B^{**} is not cyclic and is not of order 4, then we infer, as in the Remark, that $3 < m$. Let c be any element in A such that $e(c) \equiv -1$ modulo 2^m. It follows from the definition of A^* that ϕ is conformal on A^* and hence it follows from Theorem 3. 1 that A^*/A^{**} is cyclic, since B^*/B^{**} is cyclic as a group isomorphic to a group of odd integers modulo 2^m none of which is congruent to -1 modulo 4. Thus there exists an element w in A^* which generates A^* modulo A^{**}. Since the orders of B^*/B^{**} and A^*/A^{**} are divisible by 4, the element w^2 is not in A^{**}. Denote by W the subgroup of A generated by adjoining $w^{-1}c$ to A^{**}. Then $e(w^{-1}c) \equiv e(w)e(c) \equiv -e(w)$

modulo 2^m. Since w^2 is not in A^{**}, since $e(w) \equiv 1$ modulo 4, and since ϕ is conformal on A^*, it follows that $e(w) \not\equiv 1$ modulo 2^{m-1}. Since $e(w^{-1}c) \equiv -1$ modulo 4, and since $e(w^{-1}c) \not\equiv -1$ modulo 2^{m-1}, it follows from Corollary 1.2, (iii) that $(w^{-1}c)^{\phi}$ is not of order 2 modulo B^{**}. Hence W^{ϕ}/B^{**} and W/A^{**} are not of order 2 (are of an order divisible by 4); and we deduce from (a) the impossibility of W^{ϕ}/B^{**} being cyclic. This implies the existence of an element u in W^{ϕ} such that $e(u) \equiv -1$ modulo 2^m; and from $e(u) \equiv e(c^{\phi})$ modulo 2^m and from $B^{**} \leq W^{\phi}$ we deduce that c^{ϕ} is in W^{ϕ}. Hence c is in $W = A^{**}\{w^{-1}c\}$ and there exists an integer i and an element r in A^{**} such that $w = (w^{-1}c)^i r$ or $w^{1+i} = c^i r$. Since w and r are in A^{**}, since odd powers of c are not in A^{**}, it follows that i is even, $1 + i$ is odd and consequently w is in the subgroup of A which is obtained by adjoining c^2 to A^{**}. Since c^2 is in A^*, and since A^* is generated by adjoining w to A^{**}, it follows that A^* is generated by adjoining c^2 to A^{**}. Hence A is generated by adjoining c to A^{**}, since c is not in A^*, though the index of A^* in A is by Lemma 8.1 exactly 2.

If j is any integer, then we have, by Theorem 1.1,

$$\begin{aligned} e(c^{-2j}) &= e(c^{-(1+2j)}c) = e(c^{1+2j})e(c) = e(c)e(c^{1+2j}) \\ &= e(c^{e(c^{1+2j})}c^{1+2j}) = e(c^{-e(c^{-2j})+1+2j}) \end{aligned}$$

so that $e(c^{-2j}) \equiv 1 + 4j$ modulo 2^{n+1} where 2^n is the order of the cyclic groups B^*/B^{**} and A^*/A^{**}, since then 2^{n+1} is the order of c modulo A^{**}. This shows, in particular, that $e(c^2) \equiv 1 - 4$ modulo 2^n; and on the other hand it is well known that $e(c^2) \equiv 1 + 2^{m-n}e_0$ modulo 2^m for odd e_0 and $1 < n < m$. From $e(c^2) = e(c^{-1})e(c) = -e(c^{-1})$ we infer now that

$$e(c^2)^2 = e(c^{-1})^2 = e(c^{-e(c^{-1})^{-1}}) = e(c^{e(c^2)^{-1}}) = e(c^{1+2j})$$

Since ϕ is, by Lemma 8.1, conformal on A^{**}, it follows that $m - n - 1 = 1$ or $n = m - 2$. Thus 2^{m-1} is the order of c modulo A^{**} and therefore the order of A/A^{**}; and this completes the proof of the lemma.

We shall need a partial converse of this lemma.

LEMMA 8.3. *If the maximum order of the elements in the abelian 2-group A is 2^m, $(3 < m)$, if R and S are subgroups of A and if g is an element in A such that A is obtained by adjoining g to R and S is obtained by adjoining g^2 to R, if 2^{m-1} is the order of the cyclic group A/R, if $e(x)$ is a single valued S to I^*_A function such that*

(i) $e(x) = 1$ *for x in R,*

(ii) $e(g^{2j}) \equiv 1 - 4j$ *modulo 2^{m-1},*

(iii) $e(u)e(v) = e(u^{e(v)}v)$ *for u, v in S.*

then there exists a single valued A to I^*_A function $E(x)$ such that

(I) $E(x) = e(x)$ *for* x *in* S,

(II) $E(g) = -1$,

(III) $E(u)E(v) = E(u^{E(v)}v)$ *for* u, v *in* A.

Remark. It may be readily seen that R is the set of all the elements x in S satisfying: $e(x) = 1$; and that there exists at most one function $E(x)$ meeting the requirements (I), (II), (III).

Proof. The element x in A is not contained in S if, and only if, xg^{-1} belongs to S. Consequently a single valued A to I^*_A function $E(x)$ is defined by the following rule:

$$E(x) = \begin{cases} e(x) & \text{for } x \text{ in } S, \\ -e(x^{-1}g) & \text{for } x \text{ not in } S. \end{cases}$$

It is obvious that (I) and (II) are satisfied by this function $E(x)$.

Suppose that u and v are elements in A. If both u and v are in S, then (III) is a consequence of (iii); and thus we assume that not both u and v are in S.

Case 1. u is in S, v is not in S.

Then there exist elements u', v' in R and integers i, j such that $u = u'g^{2i}$, $v = v'g^{2j+1}$: Consequently

$$\begin{aligned} E(u)E(v) &= -e(u)e(v'^{-1}g^{-2j}) = -e(g^{2i})e(g^{-2j}) = -e(g^{2ie(g^{-2j})-2j}) \\ &= E(g^{1+2j-2ie(g^{-2j})}) = E(g^{2iE(g^{2j+1})}g^{2j+1}) = E(u^{E(v)}v), \end{aligned}$$

since the value of $E(x)$ depends on the coset of x modulo R only.

Case 2. u is not in S, v is in S.

Then there exist elements u', v' in R and integers i, j such that $u = u'g^{2i+1}$, $v = v'g^{2j}$. Consequently

$$\begin{aligned} E(u)E(v) &= -e(g^{-2i})e(g^{2j}) = -e(g^{-2ie(g^{2j})+2j}) \\ &= E(g^{1-2j+2ie(g^{2j})}) = E(g^{1-4j+2j+2ie(g^{2j})}) \\ &= E(g^{e(g^{2j})+2j+2ie(g^{2j})}) \text{ by (ii) and the fact that } g^{2^{m-1}} \text{ is in } R, \\ &= E(g^{(1+2i)E(g^{2j})}g^{2j}) = E(u^{E(v)}v). \end{aligned}$$

Case 3. Neither u nor v is in S.

Then there exist elements u', v' in R and integers i, j such that $u = u'g^{2i+1}$, $v = v'g^{2j+1}$. Consequently

$$\begin{aligned} E(u)E(v) &= e(g^{-2i})e(g^{-2j}) = e(g^{-2ie(g^{-2j})-2j}) \\ &= e(g^{-2ie(g^{-2j})4j-1}g^{2j+1}) = e(g^{-2ie(g^{-2j})-e(g^{-2j})}g^{2j+1}) \\ &= e(g^{-e(g^{-2j})(2i+1)}g^{2j+1}) = E(g^{(2i+1)E(g^{2j+1})}g^{2j+1}) = E(u^{E(v)}v). \end{aligned}$$

Thus condition (III) is satisfied by $E(x)$ too; and this completes the proof.

COROLLARY 8. 4. *If A, R, S, g and $e(x)$ meet the requirements of Lemma 8. 3 then* $e(g^2) \equiv -3 + k2^{m-1}$ modulo 2^m *where k is* 0 *or* 1, *and*

$$e(g^{4i}) \equiv 1 - 8i \text{ modulo } 2^m, \quad e(g^{4i+2}) \equiv -3 - 8i + k2^{m-1} \text{ modulo } 2^m.$$

Proof. Since $j(1-2j) \equiv j'(1-2j')$ modulo 2^m if, and only if, $j \equiv j'$ modulo 2^m, there exists to every given i an integer j such that $i \equiv j(1-2j)$ modulo 2^m. Consequently we find that

$$\begin{aligned} 1 - 8i &\equiv 1 - 8j(1-2j) \equiv (1-4j)^2 \equiv (1 - 4j + k_j 2^{m-1})^2 \\ &\equiv e(g^{2j})^2 \equiv e(g^{2j(1+e(g^{2j}))}) \equiv e(g^{2j(1-4j+k_j 2^{m-1}+1)}) \\ &\equiv e(g^{4j(1-2j)}) \equiv e(g^{4i}) \text{ modulo } 2^m; \text{ and consequently,} \end{aligned}$$

$$\begin{aligned} e(g^{4i+2}) &\equiv e(g^{4ie(g^2)^{-1}})e(g^2) \equiv (1 - 8ie(g^2)^{-1})e(g^2) \\ &\equiv e(g^2) - 8i \equiv -3 + k2^{m-1} - 8i \text{ modulo } 2^m. \end{aligned}$$

THEOREM 8. 5. *If A is an abelian 2-group, if 2^m is the (finite) maximum order of the elements in A, and if $e(x)$ is a single-valued A to I^*_A function, then the following conditions are necessary and sufficient for the existence of a not conformal $e(x)$-isomorphism of A upon some group B:*

(1) *If A^* is the set of all the elements x in A such that $e(x) \equiv 1$ modulo 4 and A^{**} is the set of all the elements y such that $e(y) \equiv 1$ modulo 2^m, then A^* and A^{**} are subgroups of A, A^* is of index 2 in A and A^*/A^{**} is a cyclic group.*

(2) *$e(x) \equiv e(y)$ modulo 2^m if, and only if, $x \equiv y$ modulo A^{**}.*

(3′) *If A/A^{**} does not contain elements of order 4, then $e(x)$ is a homomorphism of A (and an isomorphism of A/A^{**}).*

(3″) *If A/A^{**} contains elements of order 4, then there exists an element z in A with the following properties:*

(i) *A is generated modulo A^{**} by z.*

(ii′) *If A/A^{**} is of order 4, then*

$$e(z) \equiv -1, \quad e(z^2) \equiv 1 + 2^{m-1}, \quad e(z^3) \equiv -1 + 2^{m-1} \text{ modulo } 2^m.$$

(ii″) *If A/A^{**} is not of order 4, then there exists an integer $k = 0, 1$ such that for every integer i*

$$\begin{aligned} e(z^{2i+1}) &\equiv -e(z^{-2i}) \text{ modulo } 2^m, \\ e(z^{4i+2}) &\equiv -3 - 8i + k2^{m-1} \text{ modulo } 2^m, \\ e(z^{4i}) &\equiv 1 - 8i \text{ modulo } 2^m; \end{aligned}$$

and A/A^{**} is of order 2^{m-1}.

Proof. A. We suppose first that there exists a not conformal $e(x)$-isomorphism γ of the group A upon some group B. Then we deduce the necessity of condition (1) from Lemma 8. 1. We infer from Corollary 1. 2 the necessity of condition (2).

If A/A^{**} does not contain elements of order 4, and if x is an element not in A^{**}, then $x^{e(x)-1}$ is not in A^{**}, but x^2 is in A^{**} and $x \equiv x^{e(x)-1}$ modulo A^{**}. Hence we deduce from (2) and Theorem 1. 1 that $1 \equiv e(x^2) = e(x^{e(x)-1}x) = e(x)^2$; and now we may infer from (2) that $e(x)$ is a homomorphism of A, since it follows from (1) that A/A^{**} is either cyclic of order 2 or a direct product of two cyclic groups of order 2.

If A/A^{**} is of order 4 and contains elements of order 4, then we infer from (1) that both A/A^* and A^*/A^{**} are of order 2; and it follows from Theorem 1. 3 that $A^{**\gamma}$ and $A^{*\gamma}$ are subgroups of B such that $B/A^{*\gamma}$ and $A^{*\gamma}/A^{**\gamma}$ are of order 2. Hence it follows from Lemma 8. 2, (a) that $B/A^{**\phi}$ is not cyclic and we infer from Corollary 1. 2, (iii) the existence of an element z in A such that $e(z) = -1$. Clearly z is not in A^*. Since A/A^{**} is cyclic of order 4, it follows now that z generates A modulo A^{**}. Consequently A^* is generated modulo A^{**} by z^2. Theorem 3. 4 shows that γ is conformal on A^*. Hence $e(z^2) \equiv 1 + 2^{m-1}$ modulo 2^m; and it follows from Theorem 1. 1 that $e(z^{-1}) = e(z^{-2}z) = e(z^{2e(z)}z) \equiv e(z^2)e(z) \equiv -1 - 2^{m-1} \equiv -1 + 2^{m-1}$ modulo 2^m. This shows the necessity of (3″), (i) and (ii′). If the order of A/A^{**} is divisible by 8, then the necessity of (3″), (i) and (ii″) may be infered from Lemma 8. 3 and Corollary 8. 4.

B. We assume, conversely, that the conditions (1) to (3) are satisfied by A and $e(x)$.

B. 1. A/A^{**} does not contain elements of order 4.

Then A/A^{**} is either a cyclic group of order 2 or the direct product of two cyclic groups of order 2, as follows from (1); and we infer from (2), (3′) that $e(x) \equiv \pm 1$ or $\equiv \pm 1 + 2^{m-1}$ modulo 2^m and that $x \equiv x^{e(y)}$ modulo A^{**} for x, y in A. Hence it follows from (2), (3′) that $e(x^{e(y)}y) \equiv e(xy) \equiv e(x)e(y)$ modulo 2^m. The existence of a not conformal $e(x)$-isomorphism of A upon some group B is now a direct consequence of Theorems 1. 1 and 3. 4.

B. 2. A/A^{**} is cyclic of order 4.

In this case we deduce by direct computation from (2) and (3″), (i), (ii′) that $e(x)e(y) = e(x^{e(v)}x)$ for x, y in A; and it follows from Theorems 1. 1, 3. 4 that there exists a not-conformal $e(x)$-isomorphism of A upon some group B.

B. 3. The order of A/A^{**} is divisible by 8.

It is a consequence of (3″), (i) that there exists an element z generating

A modulo A^{**} such that $e(z^i)$ has the values given in (ii″); and it is a consequence of (2) that $e(x) = e(z^i)$ where i has been determined in such a way that $x \equiv z^i$ modulo A^{**}. If x is in A^*, then $x \equiv z^{2i}$ modulo A^{**}. Hence we deduce from (ii″) that $e(x) \equiv 1 - 4i + ki2^{m-1}$ modulo 2^m. If $y \equiv z^{2j}$ modulo A^{**}, then

$$\begin{aligned} e(x)e(y) &\equiv (1 - 4i + ki2^{m-1})(1 - 4j + kj2^{m-1}) \\ &\equiv 1 - 4(i+j) + k(i+j)2^{m-1} + 4^2 ij \\ &\equiv 1 - 4(i+j-4ij) + k(i+j-4ij)2^{m-1} \\ &\equiv e(z^{2(i+j-4ij)}) \equiv e(z^{2i(1-4j+kj2^{m-1})+2j}) \\ &\equiv e(x^{1-4j+kj2^{m-1}}y) \equiv e(x^{e(y)}y) \text{ modulo } 2^m. \end{aligned}$$

The single valued A^* to I^*_A function $e(x)$ satisfies therefore the conditions (i) to (iii) of Lemma 8.3; and hence it is readily inferred from Lemma 8.3 that the single valued A to I^*_A function $e(x)$ under consideration satisfies condition (a) of Theorem 1.1. The existence of a not-conformal $e(x)$-isomorphism of A upon some group B is now an immediate consequence of Theorems 1.1 and 3.4.

On the basis of the preceding theorem it is possible to enumerate the 2-groups that are non-conformal crossed isomorphic images of abelian 2-groups the maximum order of whose elements is 2^m. To do this we shall give a list of types of groups. There is some overlapping between these types which we are not going to investigate.

Each of the groups G which we are going to enumerate now is obtained by adjoining one or two elements to an abelian group N. This group N will always be a normal subgroup of G and G/N will be abelian too. In the following list of types N will always stand for an abelian 2-group the maximum order of whose elements is 2^{m-1} or 2^m. We use the notation: $[x, y] = x^{-1}y^{-1}xy$.

Type A. $G = \{N, g\}$, $g^2 = 1$, $g^{-1}xg = x^{-1}$ for x in N.

Type B. $G = \{N, g\}$, g^2 is in $N^{2^{m-2}}$, $g^{-1}xg = x^{-1+2^{m-1}}$ for x in N. Not both g^2 and $N^{2^{m-1}}$ are equal to 1.

Type C. $G = \{N, g, v\}$;

$v^2 = 1$, $g^{2^m} = 1$, g^2 is in N,

$v^{-1}xv = x^{-1}$, $g^{-1}xg = x^{1+2^{m-1}}$ for x *in* N, $[v, g] \equiv g^2$ modulo $N^{2^{m-2}}$.

Not all three $g^{2^{m-1}}$, $[v, g]^2 g^{-4}$ and $N^{2^{m-1}}$ are equal to 1.

Type D. $G = \{N, g, v\}$;

$v^2 = 1$, $g^{2^{m-1}} = 1$, g^2 is in N and g is not in N,

$v^{-1}xv = x^{-1}$, $g^{-1}xg = x^{1+2^{m-1}}$ for x in N, $[v, g] = g^{2+2^{m-2}}$.

Not both $g^{2^{m-2}}$ and $N^{2^{m-1}}$ are equal to 1.

Type Ek. $G = \{N, g, v\}$; $k = 0$ or 1;

$$v^2 = 1,\ g^{2^{m-1}} = 1,\ g^{2^{m-2}} \text{ is in } N \text{ and } g^{2^{m-3}} \text{ is not in } N,$$
$$v^{-1}xv = x^{-1},\ g^{-1}xg = x^{-3+k2^{m-1}} \text{ for } x \text{ in } N,\ [v, g] = g^{k2^{m-2}}.$$

Not both $g^{2^{m-2}}$ and $N^{2^{m-1}}$ are equal to 1.

THEOREM 8. 6. *The 2-group G belongs to one of the types A, B, C, D, Ek if, and only if, there exists an abelian 2-group A the maximum order of whose elements is 2^m and a not conformal crossed isomorphism of A upon G.*

Proof. Suppose first that γ is a not conformal integral $e(x)$-isomorphism of the abelian 2-group A upon the group B and that the maximum order of the elements in A is 2^m. In accordance with Theorem 8. 5 we denote by A^{**} the subgroup of all those elements in A which are mapped upon 1 by $e(x)$. It is a consequence of Theorem 8. 5 that A/A^{**} is either cyclic or a direct product of a cyclic group by a group of order 2; and that $e(x) = e(y)$ if, and only if, $x \equiv y$ modulo A^{**}. Since γ is not conformal, $1 < m$ and $e(x) \equiv -1$ modulo 4 for some x in A.

We note that γ effects an isomorphism of A^{**} upon $B^{**} = A^{**\gamma}$, that B^{**} is a normal subgroup of B, that $e(x)$ induces, by Corollary 1. 2, a homomorphism of B and an isomorphism of B/B^{**}, and that $y^{-1}xy = x^{e(y)}$ for y in B and x in B^{**} by Corollary 1. 7.

Case 1. A/A^{**} is of order 2 and $e(x) \equiv -1$ modulo 2^m for x not in A^{**}.

Then B belongs obviously to Type A.

Case 2. A/A^{**} is of order 2 and $e(x) \not\equiv -1$ modulo 2^m for x not in A^{**}.

Then we infer from Theorem 8. 5, (3′) that $e(x) \equiv -1 + 2^{m-1}$ modulo 2^m for x not in A^{**}. If $y = x^\gamma$ for some x not in A^{**}, then it follows that $y^2 = (x^{1+e(x)})^\gamma = (x^{2^{m-1}})^\gamma = ((x^2)^\gamma)^{2^{m-2}}$, since x^2 is in A^{**} and γ an isomorphism on A^{**}. If $y^2 = 1$, then $x^{2^{m-1}} = 1$ and the maximum order in A^{**} as well as in B^{**} is 2^m, since the maximum order in A is 2^m. Hence B belongs to Type B.

Case 3. A/A^{**} is of order 4, but does not contain elements of order 4.

Then A/A^{**} is the direct product of two cyclic groups of order 2 and we infer from Theorem 8. 5, (3′) the existence of elements y, z in A such that

$$e(z) \equiv -1,\ e(y) \equiv 1 + 2^{m-1},\ e(yz) \equiv -1 + 2^{m-1} \text{ modulo } 2^m.$$

The elements z, y, yz and 1 form a complete set of representatives of the cosets of A modulo A^{**}. Both z^2 and y^2 are in A^{**}. We put $v = z^\gamma$ and $w = y^\gamma$. Clearly $v^2 = 1$ and $w^2 = (y^{1+e(y)})^\gamma = (y^{2+2^{m-1}})^\gamma = ((y^2)^\gamma)^{1+2^{m-2}}$. Since y^2 is

in A^{**}, since γ is an isomorphism on A^{**}, it is now readily seen that w^2 is in B^{**} and $w^{2^m} = 1$.

From $e(yz) \equiv -1 + 2^{m-1}$ modulo 2^m and from the computation effected under Case 2 we infer that $(vw)^2$ is in $B^{**2^{m-2}}$, since vw and $y^\gamma z^\gamma$ are in the same coset modulo B^{**}. Hence

$$[v, w] = v^{-1}w^{-1}vw = vw^{-2}wvw = w^2(vw)^2 \equiv w^2 \text{ modulo } B^{**2^{m-2}};$$

and now it is readily verified that B belongs to Type C.

Case 4. A/A^{**} is cyclic of order 4.

Then we infer from Theorem 8. 5, (3″), (i), (ii′) the existence of an element z in A which generates A modulo A^{**} and meets the following requirements:

$$e(z) \equiv -1,\ e(z^2) \equiv 1 + 2^{m-1},\ e(z^3) \equiv -1 + 2^{m-1} \text{ modulo } 2^m.$$

We put $z^\gamma = v$ and $(z^2)^\gamma = w$.

Then $v^2 = 1$ and $w^2 = (z^{2(1+2^{m-1})+2})^\gamma = (z^4)^\gamma$. Consequently w^2 is in B^{**} and $w^{2^{m-1}} = 1$, since z^4 is in B^{**} and $z^{2^m} = 1$. Furthermore

$$\begin{aligned}[v, w] &= v^{-1}w^{-1}vw = vw^{-2}wvw = w^2vwvw = (z^4)^\gamma z^\gamma (z^2)^\gamma z^\gamma (z^2)^\gamma \\ &= (z^{-4+1})^\gamma (z^2)^\gamma z^\gamma (z^2)^\gamma = (z^{-3(1+2^{m-1})+2})^\gamma z^\gamma (z^2)^\gamma \\ &= (z^{1+2^{m-1}+1})^\gamma (z^2)^\gamma = (z^{(2+2^{m-1})(1+2^{m-1})+2})^\gamma = (z^{4+2^{m-1}})^\gamma \\ &= ((z^4)^\gamma)^{1+2^{m-3}} = w^{2+2^{m-2}},\end{aligned}$$

since $2 < m$, z^4 in A^{**} and γ an isomorphism in A^{**}, and since $w^2 = (z^4)^\gamma$.

If $g^{2^{m-2}} = 1$, then $z^{2^{m-1}} = 1$ so that $A^{**2^{m-1}}$ cannot be 1, since the maximum order of the elements in A is 2^m; and this shows that B belongs to Type D.

Case 5. The order of A/A^{**} is divisible by 8.

Then we deduce from Theorem 8. 5, (3″), (i), (ii″) the existence of an element z in A which generates A modulo A^{**} and whose order modulo A^{**} is 2^{m-1} and of an integer k which may be 0 or 1 subject to the following conditions:

$$e(z) \equiv -1,\ e(z^2) \equiv -3 + k2^{m-1} \text{ modulo } 2^m;$$

and we put $v = z^\gamma$, $w = (z^2)^\gamma$.

Clearly $v^2 = 1$. Since the function $e(x)$ effects an isomorphism of B/B^{**}, and since $e(w) \equiv e(z^2) \equiv 1 - 4 + k2^{m-1}$ modulo 2^m, one verifies readily that $e(w^{2^{m-3}}) \equiv 1 + 2^{m-1}$ modulo 2^m, that therefore $w^{2^{m-3}}$ is not in B^{**}, whereas $w^{2^{m-2}}$ is in B^{**}; $w^{2^{m-1}} = (z^{2^m})^\gamma = 1$. We note for future reference that $w^{2^{m-2}} = (z^{2^{m-1}})^\gamma$.

From $1 = (z^{-2}z^2)^\gamma = (z^{-2(-3)^{-1}(-3+k2^{m-1})}z^2)^\gamma = (z^{2/3})^\gamma (z^2)^\gamma$ we infer that $w^{-1} = (z^{2/3})^\gamma$. Consequently we have

$$\begin{aligned}
[v, w] &= v^{-1}w^{-1}vw = vw^{-1}vw = z^{\gamma}(z^{2/3})^{\gamma}z^{\gamma}(z^{2})^{\gamma} \\
&= (z^{e(w)^{-1}+2/3})^{\gamma}z^{\gamma}(z^{2})^{\gamma} = (z^{-e(w)^{-1}-2/3+1})^{\gamma}(z^{2})^{\gamma} \\
&= (z^{(1-e(w)^{-1}-2/3)e(w)+2})^{\gamma} \\
&= (z^{-3+k2^{m-1}-1+2+2})^{\gamma} = (z^{k2^{m-1}})^{\gamma} = ((z^{2})^{\gamma})^{k2^{m-2}} = w^{k2^{m-2}}
\end{aligned}$$

and now it is readily seen that B belongs to Type Ek.

This completes the proof of the necessity of the condition.

To prove the converse we follow closely the discussion in Cases 1 to 5, in order to construct the desired group A and crossed isomorphism γ.

If the group G belongs to Type A, then we imbed the abelian 2-group N into an abelian 2-group A such that A/N is of order 2 and such that the maximum order of the elements in A is 2^m. The function $e(x)$ is 1 in N and congruent to -1 modulo 2^m for x in A though not in N; and there exists clearly an $e(x)$-isomorphism of A upon G which is not conformal, since $1 < m$.

If the group G belongs to Type B, then there exists an element t in N such that $t^{2^{m-2}} = g^2$. Let A be an abelian group, obtained by adjoining to N an element a such that $a^2 = t$. Clearly A is an abelian group the maximum order of whose elements is 2^m. Put $e(x) = 1$ for x in N and $e(x) \equiv -1 + 2^{m-1}$, modulo 2^m, for x in A, not in N. Then we infer from Theorem 8.5 and from the results of Case 2 the existence of an $e(x)$-isomorphism of A upon G.

If G belongs to Type C, then there exists an abelian 2-group A which contains N such that A/N is the direct product of two cyclic groups of order 2 and such that A contains an element y, not in N, satisfying $y^{2+2^{m-1}} = g^2$ and an element z not in $\{N, y\}$ such that $[v, g] = g^2z^{2^{m-2}}$. There exists a homomorphism $e(x)$ of A into I^*_A which maps N upon 1, the element y upon $1 + 2^{m-1}$ and z upon -1 modulo 2^m. Then one deduces from Theorem 8.5 and from the results of Case 3 the existence of a not conformal $e(x)$-isomorphism of A upon G.

If G belongs to Type D, then there exists an abelian 2-group A the maximum order of whose elements is 2^m, which contains N and which is generated modulo N by an element z, satisfying: $z^4 = g^2$. Let $e(x) \equiv 1, -1, 1 + 2^{m-1}, -1 + 2^{m-1}$ modulo 2^m, according to x being in N, Nz, Nz^2, Nz^3. Then we infer from Theorem 8.5 and the results of Case 4 the existence of a not conformal $e(x)$-isomorphism of A upon G.

If G belongs to Type Ek for $k = 0, 1$, then there exists an abelian 2-group A the maximum order of whose elements is 2^m, which contains N and which is generated modulo N by an element z, meeting the following requirements: the order of z modulo N is 2^{m-1}, $z^{2^{m-1}} = g^{2^{m-2}}$. There exists furthermore a function $e(x)$ of the elements in A such that for y in N we have: $e(yz^{4i})$

$\equiv 1 - 8i$, $e(yz^{4i+2}) \equiv -3 - 8i + k2^{m-1}$, $e(yz^{2i+1}) \equiv -e(z^{-2i})$ modulo 2^m; and we infer from Theorem 8. 5 and from the results of Case 5 the existence of a not conformal $e(x)$-isomorphism of A upon G. This completes the proof of the Theorem.

9. Structure of composite groups, obtained by integral crossed isomorphisms. If ϕ is an integral crossed isomorphism of the abelian group A upon the group B, then there exists a uniquely determined single valued A to I^*_A function $e(x)$ such that ϕ is an $e(x)$-isomorphism. If A contains elements of order 0, then $e(x) = \pm 1$ for every x in A; and the nature of B has been discussed completely in **2**.

Thus we assume henceforth that A does not contain elements of order 0; and this is the case if, and only if, B does not contain elements of order 0. Under these circumstances the group A is known to be the direct product of its primary components A_p (where A_p is the subgroup of all the elements of order a power of p in A). It is readily seen that B is the product of its p-components $B_p = A_p^{\phi}$, though B_p need not be the only p-component of B. Finally it will be convenient to denote by A_p^+ the subgroup of A which consists of all the elements in A whose order is prime to p. Then A is the direct product of A_p and A_p^+ and B is the product of B_p and $B_p^+ = A_p^{+\phi}$.

LEMMA 9. 1. *If p and q are different prime numbers, if q is not a divisor of $p-1$, if A is a cyclic group of order p^iq^j, and if ϕ is a crossed isomorphism of A upon the group B, then B_q is a normal and cyclic subgroup of B.*

Proof. It is a consequence of our hypothesis that q is odd. Hence it follows from Theorem 3. 4 that ϕ is conformal on A_q showing that B_q is a cyclic group of order q^j. There exists a uniquely determined single valued A to I^*_A function $e(x)$ such that ϕ is an $e(x)$-isomorphism. We denote by E the subset of all the elements x in A such that $e(x) = 1$; and we put $N = E^{\phi}$. It is a consequence of Corollary 1. 7 that E is a subgroup of A on which ϕ is an ordinary isomorphism, that N is an abelian, normal subgroup of B and that B/N is abelian too.

The p-component N_p of N is a cyclic characteristic subgroup of N and its order is a divisor of p^i. If g is any element in B_q, then g transforms N, and therefore N_p, into itself. Thus g induces in N_p an automorphism of order a power of q. Since the order of N_p is a divisor of p^i, and since N_p is cyclic, the order of every automorphism of N_p is a divisor of $p^{i-1}(p-1)$. But no power of q with the exception of $q^0 = 1$ is a divisor of $p^{i-1}(p-1)$. This shows that every element in B_q commutes with every element in N_p. Since N is the direct product of the cyclic groups N_p and $N_q \leq B_q$, it follows that NB_q is an abelian group and that it is the direct product of N_p and B_q.

From the commutativity of B/N we infer that NB_q is a normal subgroup of B. Since B_q has been shown to be the q-component of the abelian normal subgroup NB_q of B, it follows finally that B_q is a normal subgroup of B.

COROLLARY 9.2. *Suppose that the abelian group A without elements of order 0 is the direct product of groups R and S meeting the following requirement:*

(9.2.1) *If the prime numbers p and q are orders of elements in R and S respectively, then $p \neq q$ and q is not a divisor of $p-1$.*

Then every integral crossed isomorphism ϕ of A upon a group B maps S upon a normal subgroup of B.

Proof. It is a consequence of Theorem 1.3 that R^ϕ and S^ϕ are subgroups of B. It is readily verified that every element r in R^ϕ has the form: Πr_i^ϕ where the r_i are elements of prime power order in R, that every element s in S^ϕ has the form Πs_i^ϕ where the s_i are elements of prime power order in S, and that every element x in B has the form: $x = x'x''$ for x' in R^ϕ, x'' in S^ϕ. If y is an element in S^ϕ, x an element in B, then $xyx^{-1} = x'(x''yx''^{-1})x'^{-1}$ where $s = x''yx''^{-1}$ is in S^ϕ. From a previous remark we infer the existence of elements r_i, s_i of prime power order in the groups R and S respectively such that $s = s_1^\phi \cdots s_k^\phi$, $x' = r_1^\phi \cdots r_h^\phi$. From (9.2.1) and Lemma 9.1 it follows that r_i^ϕ transforms the cyclic group generated by s_j^ϕ into itself; and this shows that S^ϕ is a normal subgroup of B.

COROLLARY 9.3. *If A is an abelian group without elements of order 0, if $p_1 < p_2 < \cdots < p_i < \cdots$ are the prime numbers that are orders of elements in A, if A_i is the subgroup of all the elements in A whose orders are prime to $p_1 \cdots p_{i-1}$, and if ϕ is a crossed isomorphism of A upon a group B, then A_i^ϕ is a normal subgroup of B.*

This is an immediate consequence of Corollary 9.2, since $p < q$ implies $p \neq q$ and that q is not a divisor of $p-1$.

LEMMA 9.4. *If p is an odd prime number, if the p-group B is the integral crossed isomorphic map of an abelian group, if there exists an integral automorphism of B whose order is a prime number, not p, then B is abelian.*

Proof. Suppose that the integral crossed isomorphism ϕ maps the abelian group A upon B. Then we infer from Theorem 3.5 that A is a p-group and from Theorem 3.4 that ϕ is conformal. It is a consequence of Theorem 7.1 that B is abelian, if the orders of the elements in A and B are not bounded. Thus we assume that the orders of the elements in B are bounded. Then we infer from Theorem 7.1 the existence of a normal abelian subgroup N of B

such that B/N is cyclic and such that every element in B induces an integral automorphism in N. We denote by g some element which generates B modulo N. Then there exists an integer e, prime to p, such that $g^{-1}xg = x^e$ for x in N.

Let α be an automorphism of B whose order is a prime number $q \neq p$ and which satisfies: $S = S^\alpha$ for every subgroup S of B. We denote by F the subgroup of all the fixed elements under the automorphism α. From $\alpha \neq 1$ we infer $F \neq B$. Hence the coset Ng cannot be part of F, since otherwise F would contain both g and N and therefore B. Since all the elements in Ng induce the same automorphism in the abelian group N, we may assume without loss in generality that g does not belong to F. Hence there exists an integer k, prime to p, such that $g^k = g^\alpha \neq g$. Since g is of order a power of p, since α is of order q, it follows that $g = g^{k^q}$. Hence we deduce from well known theorems on the automorphisms of cyclic groups that both k and $k-1$ are prime to p. If x is an element in N, then $x = y^\alpha$ for y an element in N. Hence

$$g^{-1}xg = g^{-1}y^\alpha g = y^{\alpha e} = y^{e\alpha} = (g^{-1}yg)^\alpha = g^{-\alpha}y^\alpha g^\alpha = g^{-k}y^\alpha g^k = g^{-k}xg^k$$

or $xk^{k-1} = g^{k-1}x$ for every x in N. But g and g^{k-1} generate the same cyclic subgroup of B, showing that g itself permutes with every element in N. This proves the commutativity of B, since N is commutative, and since B is generated modulo N by g.

COROLLARY 9.5. *If p and q are different prime numbers, if q is not a divisor of $p-1$, if the abelian group A is the direct product of the p-group A_p and the q-group A_q, and if ϕ is an integral crossed isomorphism of A upon the group B, then B is either the direct product of the p-group $A_p{}^\phi = B_p$ and the q-group $A_q{}^\phi = B_q$ or else B_q is an abelian normal subgroup of B such that every element in B induces an integral automorphism in B_q.*

Proof. It is a consequence of Theorems 1.3 and 3.5 that B_p is a p-group and that B_q is a q-group. We note furthermore that B is the product of B_p and B_q. It is a consequence of Corollary 9.2 that B_q is a normal subgroup of B. If x is an element in B_p and y an element in B_q, then we infer from Lemma 9.1 that $x^{-1}yx$ generates the same subgroup as y. If B is not the direct product of B_p and B_q, then there exists an element x in B_p which induces in B_q an integral automorphism of order p. Consequently we infer from Lemma 9.4 the commutativity of B_q, so that every element in B induces an integral automorphism in the abelian group B_q, unless B is the direct product of B_p and B_q.

THEOREM 9.6. *The group G is the integral crossed isomorphic map of an abelian group without elements of order* 0 *if, and only if, G contains a*

system of primary components G_p *for* $p = 2, 3, 5, \cdots$ *with the following properties*:

(1) G_p *is the integral crossed isomorphic map of an abelian p-group.*

(2) G *is the product of the subgroups* G_p.

(3) *If* $p < q$, *then* G_p *is part of the normalizer of* G_q *in* G.

(4′) *If* $p < q$, *and if* G_q *is not abelian, then* G_p *is part of the centralizer of* G_q *in* G.

(4″) *If* $p < q$, *and if* G_q *is abelian, then every element in* G_p *induces an integral automorphism in* G_q.

Remark. The primary groups that are integral crossed isomorphic maps of primary abelian groups have been determined in **2**, **7** and **8**.

Proof. Suppose first that there exists an abelian group A without elements of order 0 and an integral crossed isomorphism κ of A upon the group G. Then A is the direct product of its primary components A_p. It is a consequence of Theorem 1. 3 that $G_p = A_p{}^\kappa$ is a subgroup of G; and G_p is a p-group by Theorem 3. 5. That these subgroups G_p satisfy condition (3) is a consequence of Corollary 9. 3; and now it is readily verified that every G_p is a p-component of G, and that the G_p satisfy (1) and (2). The necessity of conditions (4′), (4″) is an immediate consequence of Corollary 9. 5.

Assume, conversely, the existence of p-components G_p of G meeting the requirements (1) to (4). There exists by (1) to every prime number p an abelian p-group A_p and an integral crossed isomorphism $\kappa(p)$ of A_p upon G_p. There is no loss in generality in assuming that $\kappa(p)$ is an ordinary isomorphism, if G_p happens to be abelian. The direct product of the A_p will be denoted by A.

It is a consequence of conditions (2) and (3) that every element g in G may be represented in one and only one way in the form:

$$g = g(2)g(3)g(5)\cdots g(p)\cdots \text{ for } g(p) \text{ in } G_p$$

where almost every $g(p) = 1$ and where the order of the factors is essential ($g(p)$ appears before $g(q)$, if $p < q$). Likewise every element u in A may be represented in one and only one fashion in the form:

$$u = u(2)u(3)u(5)\cdots u(p)\cdots \text{ for } u(p) \text{ in } A_p$$

where almost every $u(p) = 1$. Thus a $1:1$ correspondence κ mapping the abelian group A upon the set of all the elements in G is defined by

$$u^\kappa = u(2)^{\kappa(2)}u(3)^{\kappa(3)}u(5)^{\kappa(5)}\cdots u(p)^{\kappa(p)}\cdots;$$

and κ induces in A_p the integral crossed isomorphism $\kappa(p)$.

If u is an element in A and x an element in A_p, and if G_p is not abelian, then we put $x^{e(u)} = x^{e(p;u(p))}$ where $e(p;y)$ is the uniquely determined single valued A_p to $I^*_{A_p}$ function such that $\kappa(p)$ is an $e(p;y)$-isomorphism of A_p upon G_p. If u is an element in A and x an element in A_p, and if G_p is abelian, then we put

$$x^{e(u)} = ((\prod_{q<p} u(q)^{\kappa})^{-1} x^{\kappa} \prod_{q<p} u(q)^{\kappa})^{\kappa^{-1}}.$$

It is a consequence of (4″) that this function $e(u)$ is, for every u in A, a well determined integral automorphism of A_p. Consequently there exists one and only one integral automorphism $e(u)$ of A which coincides on every A_p with the integral automorphisms $e(u)$, just defined.

If u and v are elements in A, then there exist primes $p_1 < p_2 < \cdots < p_n$ and elements u_i, v_i in A_{p_i} such that $u = \prod_i u_i$, $v = \prod_i v_i$. We deduce from conditions (4′) and (4″) that $u^{\kappa} v^{\kappa} = \prod_{i=1}^{n-1} u_i^{\kappa} \prod_{i=1}^{n-1} v_i^{\kappa} (u_n^{e(v)} v_n)^{\kappa}$; and we show by complete induction that

$$u^{\kappa} v^{\kappa} = (u^{e(v)} v)^{\kappa} \text{ for } u \text{ and } v \text{ in } A,$$

proving that κ is an integral crossed isomorphism of A upon G, as was to be shown.

THE UNIVERSITY OF ILLINOIS,
URBANA, ILLINOIS.

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A PHRAGMÉN LINDELÖF THEOREM.*

By AUDREY WISHARD MCMILLAN.

This paper presents a theorem of the Phragmén Lindelöf type. Its main result is an improvement of a theorem of F. Wolf [1]. Our theorem is:

THEOREM 1. *Suppose that*

(a) $f(z)$ *is analytic in* $x \geqq 0$,

(b) $|f(iy)| \leqq 1$,

(c) $|f(re^{i\theta})| \leqq \exp[re^{\psi(\theta)}]$ *in* $-\pi/2 < \theta < \pi/2$, *where* $\int_{-\pi/2}^{\pi/2} \psi(\theta)\,d\theta$ *exists in the sense of Lebesgue.*

Then there is a number k *such that* $|f(x+iy)| \leqq e^{kx}$. *Moreover,*

$$k \leqq \varlimsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r\cos\theta} \tag{1}$$

for any θ, $-\pi/2 < \theta < \pi/2$, *and* k *need not be positive.*

This represents an improvement over the result of F. Wolf in that he demands $|f(re^{i\theta})| \leqq \exp(\epsilon re^{\psi(\theta)})$ for every $\epsilon > 0$, for large r. (This implies that (1) holds with $k = 0$ for almost all θ). He then concludes that $|f(z)| < 1$.

The first part of our proof follows closely a proof given by Carleman [2]. The theorem which he proves differs from ours in demanding that $|f(z)| \leqq \exp(e^{\psi(\theta)})$ (with no r present) and that $f(z)$ be an entire function. Since the argument is crucial, we give it here, using Carleman's notation.

Let $\boldsymbol{C}(R)$ be the region enclosed by the half-circle: $r = R$, $-\pi/2 < \theta < \pi/2$, and let $C(R)$ be the curved part of its boundary. Let $v(r)$ be the maximum of $\log|f(z)|$ on $C(r)$.

The proof of our theorem will be simpler if we preface it by giving a short discussion of the properties of $v(r)$. We suppose that r is large enough to make $v(r) > 0$; then $\log|f(z)|$ does not take on the value $v(r)$ on the imaginary axis. Therefore $v(r)$ has the usual properties of the maximum. It is continuous and increasing. Proceeding as in the proof of the Hadamard three circle theorem [4], we let $r_1 < r_2 < r_3$ and let r_1 be large enough so that $v(r_1) > 0$ and $r_1 > 1$. In the region between $C(r_1)$, $C(r_3)$ and the line

* Received February 19, 1943.

$x=0$, consider the function $\phi(z)=z^{\lambda}f(z)$ with λ negative. $\phi(z)$ takes on its maximum on the boundary of the region, in fact either on $C(r_1)$ or $C(r_3)$ because $|f(z)|>1$ on $C(r_1)$ and $|f(z)|<1$ on $x=0$. Thus if M_j is the maximum of $|f(z)|$ on $C(r_j)$, we have,

$$r_2{}^{\lambda}M_2 \leqq \max\,[r_1{}^{\lambda}M_1, r_3{}^{\lambda}M_3]$$

and, choosing λ so that $r_1{}^{\lambda}M_1=r_3{}^{\lambda}M_3$, we find that

$$(\log r_3-\log r_1)\log M_2 \leqq (\log r_3-\log r_2)\log M_1+(\log r_2-\log r_1)\log M_3.$$

Thus

$$\frac{v(r_2)-v(r_1)}{\log r_2-\log r_1} \leqq \frac{v(r_3)-v(r_2)}{\log r_3-\log r_2}.$$

Taking the lim sup as $r_2\to r_1$ and then lim inf as $r_3\to r_1$ we see that $d\,v(r)/d\log r$ exists on the right. Similarly, it exists on the left and we have the important relation

$$\left.\frac{d\,v(r)}{d\log r}\right|_{r-} \leqq \left.\frac{d\,v(r)}{d\log r}\right|_{r+}.$$

To return to our main theorem we first prove that there is a K such that $|f(z)|\leqq e^{Kr}$. We suppose that this is not true. That is, we assume that $v(r)/r$ is unbounded as r becomes infinite. Let R be large enough so that $v(R)>2$ and let $G(R)$ be the region interior to $\boldsymbol{C}(R)$ in which $\log|f(z)|>v(R)/2$. This region has no points in common with the line $x=0$. Its boundary is made up of arcs interior to $\boldsymbol{C}(R)$ on which $\log|f(re^{i\theta})|=v(R)/2$ and of arcs $\alpha(R)$ on $C(R)$ on which $\log|f(Re^{i\theta})|\geqq v(R)/2$. Consider the function $v(R)-\log|f(re^{i\theta})|$. It is harmonic in $G(R)$, is $\geqq 0$ on the $\alpha(R)$ and is $=v(R)/2$ on the remainder of the boundary of $G(R)$. In $G(R)$ it is therefore $\geqq$ the harmonic function which is zero on the $\alpha(R)$ and is $=v(R)/2$ on the remainder, $\beta(R)$, of the full circle $r=R$. But for a properly chosen θ, $\log|f(re^{i\theta})|=v(r)$ and thus, for such θ,

$$v(R)-v(r)\geqq\frac{1}{2\pi}\int_{\beta(R)}\frac{v(R)}{2}\,\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\vartheta-\theta)}\,d\vartheta,$$

$$\lim_{r\to R}\frac{v(R)-v(r)}{R-r}\geqq\frac{v(R)}{4\pi R}\int_{\beta(R)}\frac{1}{1-\cos(\vartheta-\theta)}\,d\vartheta.$$

(It has been proved above that this limit exists.) Let $a(r)$ be the angular measure of the set $\alpha(r)$. Then

$$\left.\frac{dv(r)}{dr}\right|_R\geqq\frac{v(R)}{4\pi R}\int_{a(R)/2}^{2\pi-[a(R)/2]}\frac{1}{1-\cos\vartheta}\,d\vartheta,\qquad =\frac{v(R)}{2\pi R}\cot\frac{a(R)}{4}.$$

The proof involves the derivative on the left and holds *a fortiori* for the derivative on the right.

Now let $\lambda(r)$ be the set of values θ on $C(r)$ for which $\psi(\theta) > \log[v(r)/2r]$, and let $L(r)$ be its measure. Then $\lambda(r)$ certainly contains $\alpha(r)$, for if θ belongs to $\alpha(r)$, we have $\log|f(re^{i\theta})| \geqq v(r)/2$ and hence $v(r)/2r \leqq \exp(\psi(\theta))$. Since $L(r) < \pi$,

$$(d/dr)\log[v(r)/2]|_R \geqq (1/\pi R)\cdot(1/L(R))$$

$$(d/dr)\log[v(r)/2r]|_R \geqq 1/\pi R L(R) - 1/R. \tag{2}$$

So far R is subject only to the restriction that $v(R) > 2$, an inequality which persists for $r > R$. It must be possible to choose R so that, in addition, $L(R) < 1/\pi$, for, $v(r)/r$ being unbounded, we can choose R so that the set where $\psi(\theta) > \log[v(R)/2R]$ is arbitrarily small. Then (2) shows that $\log[v(r)/2r]$ is increasing at R. But this means that $L(r) \leqq L(R) < 1/\pi$ and that $v(r)/2r$ is increasing in some neighborhood $r_0 < r, R < r'$. Define $r_1 =$ l. u. b. of all such r'. By continuity, $v(r_1)/2r_1 > v(R)/2R$. Then $L(r_1) \leqq L(R)$, and (2), with $R = r_1$, implies a contradiction. Therefore, for all $r > R$, we have the right hand side of (2) $>$ const.$/rL(r)$ and

$$\int_R^\infty L(r)(d/dr)\log[v(r)/2r]\,dr \tag{3}$$

is divergent.

We make (3) into a proper integral by putting on an arbitrary upper limit R'. If $F(r)$ is a function depending only on r, $L(r)F(r) = \int_{\lambda(r)} F(r)\,d\theta$ and therefore our integral can be written as a double integral over the region G in which $R < r < R'$ and $\psi(\theta) > \log[v(r)/2r]$. Therefore, for an appropriate $R(\theta)$, $R \leqq R(\theta) \leqq R'$,

$$\int_R^{R'}\int_{\lambda(r)}(d/dr)\log[v(r)/2r]\,d\theta\,dr = \int_{-\pi/2}^{\pi/2}\int_R^{R(\theta)} d\log[v(r)/2r]\,d\theta \leqq \int_{-\pi/2}^{\pi/2}\psi(\theta)\,d\theta.$$

This contradicts our hypothesis that $\int_{-\pi/2}^{\pi/2}\psi(\theta)\,d\theta$ converges.

We know, now, that $|f(z)| \leqq e^{K|z|}$ and we can apply a Phragmén Lindelöf theorem of the standard sort [3]. In the angle $\theta_1 < \theta < \pi/2$, we consider the function $F(z) = e^{-(k+\epsilon)z}f(z)$, where (cf. (1))

$$k = \overline{\lim_{r\to\infty}}\,\frac{\log|f(re^{i\theta_1})|}{r\cos\theta_1}.$$

$F(z)$ is analytic and $|F(z)| \leqq e^{K|z|}$ in the angle; the opening of the angle is $< \pi$; and $|F(z)|$ is bounded on both of the boundary lines. Hence $|F(z)|$ is bounded throughout the angle. Similarly, $|F(z)|$ is bounded in $-\pi/2 < \theta < \theta_1$. But if $|F(z)|$ is bounded in $x \geqq 0$ and is $\leqq 1$ on $x = 0$, the bound is 1 for the entire half plane. Therefore $|e^{-kz}f(z)| \leqq |e^{\epsilon z}|$ for $x \geqq 0$. Letting $\epsilon \to 0$, we find $|f(z)| \leqq e^{kx}$ in $x \geqq 0$.

The method used in the proof of Theorem 1 can be exploited further. The following theorem is stronger but somewhat more complicated than the previous one and its proof is, as we shall show, almost identical.

THEOREM 2. *Suppose that*

(a) $f(z)$ *is analytic in* $x \geqq 0$,

(b′) *there is a positive non-decreasing function* $\phi(y)$ *in* $0 < y < \infty$ *such that* $\log |f(iy)| \leqq \phi(|y|) \leqq M|y|$, *for some* M, *and such that*

$$\int^{\infty} (\phi(y)/y^2)\,dy \text{ converges,}$$

(c′) $|f(re^{i\theta})| \leqq \exp(r^{\alpha}e^{\psi(\theta)})$, $-\pi/2 < \theta < \pi/2$, *where* $\int_{-\pi/2}^{\pi/2} \psi(\theta)\,d\theta < \infty$ *and* $\alpha < \pi/(\pi/2 + |\theta_1|)$ *and* θ_1 *is an angle such that*

(d) $k = \overline{\lim_{r\to\infty}} \dfrac{\log |f(re^{i\theta_1})|}{r\cos\theta_1}$ *is finite*, $|\theta_1| < \pi/2$.

Then for $|x + iy| > 1$, $|f(x+iy)| \leqq \exp[\phi(2y) + (k+\xi)x]$ *where*

$$\xi \leqq (4/\pi)\int_{-\infty}^{\infty} [\phi(t)/(1+t^2)]\,dt \quad \text{and} \quad \xi \to 0 \text{ as } (x,y) \to \infty.$$

Obviously the condition (c′) can be replaced by a somewhat simpler condition which is independent of θ_1:

(c″) $$|f(re^{i\theta})| \leqq \exp(r \log r\, e^{\psi(\theta)}).$$

Again we first assume that there is no K such that $|f(z)| \leqq e^{Kr^{\alpha}}$. Suppose that R is such that $v(R)/R > 2M$, (M given in (b′)) and also such that the measure, $L(R)$, of the set where $v(R)/2R < \psi(\theta)$ satisfies $L(R) < 1/\pi\alpha$. Then in $|y| < R$, $\phi(y) < MR < v(r)/2$, and, as before, the region $G(R)$ does not have any boundary points on $x = 0$. We proceed as before and find that

$$[d \log(v(r)/2r^{\alpha})/dr]|_R \geqq (1/\pi R)\cdot(1/L(R)) - \alpha/R.$$

Thus $v(R)/R^{\alpha}$ is increasing, $L(R)$ decreasing, and the inequality holds for all $r > R$. One continues through to the contradiction as before.

Knowing that $|f(z)| < e^{Kr^{\alpha}}$, we turn to standard Phragmén Lindelöf

methods. Let $U(z)$ be the positive harmonic function which is equal to $\phi(y)$ on $x=0$ and which increases like kx:

(4) $$U(z)=(1/\pi)\int_{-\infty}^{\infty}\phi(|t|)\frac{x}{x^2+(y-t)^2}dt+kx$$

and let $\Phi(z)$ be the analytic function of which $U(z)$ is the real part. Consider $F(z)=\exp(-\Phi(z)-\epsilon z)f(z)$. We have, since $U(z)\geqq 0$, $|F(z)|\leqq|f(z)|\leqq e^{Kr^a}$ in $x>0$ and $|F(iy)|\leqq e^{-\phi(|y|)}|f(iy)|\leqq 1$, and $\lim\limits_{r\to\infty}\log|F(re^{i\theta_1})|\leqq 0$. Thus $F(z)$ is bounded in $\theta_1<\theta<\pi/2$ as well as in $-\pi/2<\theta<\theta_1$, and since the bound is 1 on $x=0$, it is 1 throughout $x>0$. Then $|e^{-\Phi(z)}f(z)|\leqq|e^{\epsilon z}|$ and letting $\epsilon\to 0$ we have, finally,

(5) $$\log|f(z)|\leqq U(z).$$

Now suppose that $y>0$. Since $\phi(y)$ is symmetric, the following holds also for negative y when $|x+iy|>1$,

$$\int_{-\infty}^{\infty}\phi(|t|)\frac{x}{x^2+(y-t)^2}dt=\int_{-\infty}^{0}+\int_{0}^{2y}+\int_{2y}^{\infty}\phi(|t|)\frac{x}{x^2+(y-t)^2}dt$$

$$\leqq x\int_{-\infty}^{0}\phi(|t|)\frac{1}{x^2+y^2+t^2}dt+x\int_{2y}^{\infty}\phi(t)\frac{1}{x^2+(t/2)^2}dt$$

$$+\phi(2y)\int_{0}^{2y}\frac{x}{x^2+(y-t)^2}dt$$

(6) $$\leqq\pi x\xi+\phi(2y)\cdot 2\arctan(y/x)\leqq\pi x\xi+\pi\phi(2y)$$

where $\pi\xi\leqq 4\int_{-\infty}^{\infty}\phi(|t|)\frac{1}{1+t^2}dt$ and $\xi\to 0$ as $(x,y)\to\infty$.

(7) $$U(z)\leqq\xi x+\phi(2y)+kx.$$

This, combined with (5), completes the proof.

In many cases a much better inequality than (7) can be found. Thus, if $\phi(y)\leqq M$, $|f(z)|\leqq e^{M+kx}$ and if $\phi(y)\leqq M\log(1+y^2)$ we get $|f(z)|\leqq[(1+x)^2+y^2]^M e^{kx}$, this latter statement being true because, in this case, $U(z)=M\log[(1+x)^2+y^2]+kx$.

In general

(8) *if* $\phi(y)$ *satisfies* $\phi(ut)\leqq\phi(u)\phi(t)$, *then*

$$|f(z)|\leqq\exp(\phi(2y)+\xi_1\cdot\phi(x)+kx),\ \xi_1=(4/\pi)\int_{-\infty}^{\infty}[\phi(|t|)/(1+t^2)]dt.$$

In order to prove (8) we return to (6). We have for $y > 0$

$$\int_{-\infty}^{\infty} \phi(|t|) \frac{x}{x^2 + (y-t)^2} dt \leqq \int_{-\infty}^{\infty} \phi(|t|) \frac{4x}{x^2 + t^2} dt + \phi(2y) \cdot 2 \operatorname{arc\,tan} (y/x)$$

$$\leqq 4 \int_{-\infty}^{\infty} \phi(|ux|) \frac{1}{1+u^2} du + \pi\phi(2y)$$

$$\leqq 4\phi(x) \int_{-\infty}^{\infty} \phi(|u|) \frac{1}{1+u^2} du + \pi\phi(2y).$$

It is to be noted that (8) is satisfied by all functions of the form $\phi(t) = t^p$ as well as by those already mentioned. One cannot prove (8) without some such condition as is given, for if $\phi(t)$ is of the form: $\phi(t) = 0$, $0 < t < N$; $\phi(t) = M$, $t > N$, then $U(z) = (M/\pi) \operatorname{arc\,tan} [x/(N-y)]$ and $U(x)$ behaves very much like $(M/\pi N)x$ when x/N is small.

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THE NUMBER OF ABSOLUTE INVARIANTS OF A TENSOR.*

By Richard H. Bruck.

1. The number of invariants of a tensor. Let $T^{(i)}_{(j)} \equiv T^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ be a tensor with p contravariant and q covariant indices, $(p, q \geqq 0)$, in an n-dimensional coördinate system. In this paper we study the number of functionally independent absolute invariants of $T^{(i)}_{(j)}$ under the general linear group.

To elaborate the above statement, let $f(T)$ be a function of the n^{p+q} components (which may or may not be independent) of the tensor T. Let $\bar{T}$ be a tensor obtained from T by an arbitrary transformation of the general linear group, as indicated by equation (1) below. Then we say that f is an *absolute invariant* of T if and only if $f(\bar{T}) \equiv f(T)$. Again, if s functions $f_1, f_2, \cdots, f_s$ are functionally independent, are absolute invariants of T and have the property that every other invariant f is functionally dependent upon them, we say that T possesses exactly s *functionally independent absolute invariants.* Thus it is our purpose to determine s for various types of tensor T.

The tensor T transforms according to

$$(1) \qquad \bar{T}^{(i)}_{(j)} \equiv \bar{T}^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} = a_{k_1}{}^{i_1} a_{k_2}{}^{i_2} \cdots a_{k_p}{}^{i_p} \cdot \alpha_{j_1}{}^{l_1} \alpha_{j_2}{}^{l_2} \cdots \alpha_{j_q}{}^{l_q} \cdot T^{k_1 k_2 \cdots k_p}_{l_1 l_2 \cdots l_q},$$

where the $a_j{}^i$ are n^2 indeterminates and where

$$(2) \qquad a_m{}^i \alpha_j{}^m = \delta_j{}^i, \text{ the simple Kronecker delta.}$$

The identity transformation $a = a_0$ is given by

$$(3) \qquad a_j{}^i = a_{0j}{}^i \equiv \delta_j{}^i.$$

Since the $a_j{}^i$ are independent,

$$(4) \qquad \partial a_j{}^i / \partial a_\beta{}^\alpha = \delta_\alpha{}^i \delta_j{}^\beta,$$

and from (2) and (4),

$$(5) \qquad 0 = (\partial / \partial a_\beta{}^\alpha)(a_m{}^i \alpha_j{}^m) = \delta_\alpha{}^i \delta_m{}^\beta \alpha_j{}^m + a_m{}^i (\partial \alpha_j{}^m / \partial a_\beta{}^\alpha).$$

Equations (4) and (5) yield, for $a = a_0$,

* Received February 5, 1943; presented to the American Mathematical Society, September 2, 1941.

$$(6) \qquad \partial a_j{}^i/\partial a_\beta{}^\alpha]_{a=a_0} = \delta_\alpha{}^i \delta_j{}^\beta; \qquad \partial \mathfrak{a}_j{}^i/\partial a_\beta{}^\alpha]_{a=a_0} = -\,\delta_\alpha{}^i\delta_j{}^\beta.$$

If (1) be differentiated with respect to $a_\beta{}^\alpha$, then

$$(7) \quad \partial \bar{T}{}^{(i)}_{(j)}/\partial a_\beta{}^\alpha = [\partial a_{k_1}{}^{i_1}/\partial a_\beta{}^\alpha \cdots a_{k_p}{}^{i_p}\mathfrak{a}_{j_1}{}^{l_1}\cdots \mathfrak{a}_{j_q}{}^{l_q} + \cdots + a_{k_1}{}^{i_1}\cdots(\partial a_{k_p}{}^{i_p}/\partial a_\beta{}^\alpha)\mathfrak{a}_{j_1}{}^{l_1}\cdots \mathfrak{a}_{j}$$
$$+\, a_{k_1}{}^{i_1}\cdots a_{k_p}{}^{i_p}(\partial \mathfrak{a}_{j_1}{}^{l_1}/\partial a_\beta{}^\alpha)\cdots \mathfrak{a}_{j_q}{}^{l_q} + \cdots + a_{k_1}{}^{i_1}\cdots a_{k_p}{}^{i_p}\mathfrak{a}_{j_1}{}^{l_1}\cdots \partial\mathfrak{a}_{j_q}{}^{l_q}/\partial a_\beta{}^\alpha]\cdot T$$

Thus if we define

$$(8) \qquad \xi^{(i)\beta}_{(j)\alpha} = \partial \bar{T}{}^{(i)}_{(j)}/\partial a_\beta{}^\alpha]_{a=a_0},$$

we obtain from (6) and (7)

$$(9) \quad \xi^{(i)\beta}_{(j)\alpha} = [\delta_\alpha{}^{i_1}T^{\beta i_2\cdots i_p}_{(j)} + \cdots + \delta_\alpha{}^{i_p}T^{i_1 i_2\cdots\beta}_{(j)}] - [\delta_{j_1}{}^\beta T^{(i)}_{\alpha j_2\cdots j_q} + \cdots + \delta_{j_q}{}^\beta T^{(i)}_{j_1 j_2\cdots\alpha}$$

The tensor $\xi^{(i)\beta}_{(j)\alpha}$ may be displayed as a matrix $\| \xi_U{}^S \|$ in which the $p+q$ indices $S = {}^{(i)}_{(j)}$ determine the n^{p+q} rows and the two indices $U = {}^{\beta}_{\alpha}$ determine the n^2 columns. Adapting a theorem of Eisenhart [1] to the present situation we may state

THEOREM I. *The number of functionally independent absolute invariants of* $T^{(i)}_{(j)}$ *is* $N - Q$, *where* N *is the number of independent components of the tensor and* Q *is the generic*[1] *rank of the matrix* $\| \xi_U{}^S \|$.

On the other hand, the number of linearly independent tensors $F_j{}^i$ satisfying $\xi^{(i)\beta}_{(j)\alpha}F_\beta{}^\alpha = 0$ is clearly $n^2 - Q$; hence the result may be put in the following tensorial form:

THEOREM I′. *The number of functionally independent absolute invariants of a tensor* $T^{(i)}_{(j)} \equiv T^{i_1 i_2\cdots i_p}_{j_1 j_2\cdots j_q}$ *is equal to* $N - Q$. *Here* N *is the number of independent components of* $T^{(i)}_{(j)}$ *and* $n^2 - Q$ *is the generic number of linearly independent tensors* $F_j{}^i$ *which satisfy the tensor equation* $\xi^{(i)\beta}_{(j)\alpha}F_\beta{}^\alpha = 0$ *or the equivalent equation*

$$(10) \quad F_\beta{}^{i_1}T^{\beta i_2\cdots i_p}_{(j)} + \cdots + F_\beta{}^{i_p}T^{i_1 i_2\cdots\beta}_{(j)} = F_{j_1}{}^\alpha T^{(i)}_{\alpha j_2\cdots j_q} + \cdots + F_{j_q}{}^\alpha T^{(i)}_{j_1 j_2\cdots\alpha}.$$

In deducing the above theorems from the theory of continuous groups we have tacitly assumed that the underlying field K is either the field of all reals or the field of all complex numbers. Nevertheless, the numerical in-

[1] The word generic has roughly the same force as the phrase *in general* which was employed so frequently by the English geometers of the nineteenth century. Thus Q is the rank of $|| \xi_U{}^S ||$ when T is an arbitrary (or unspecialized) tensor of whatever symmetry type we have under consideration.

variants N and Q exist for any K and remain unchanged under extensions of this field. In the following treatment we shall assume merely that K is non-modular.

Corresponding to any permutation $\pi = \begin{pmatrix} 1 & 2 & \cdots & p \\ \pi_1 & \pi_2 & \cdots & \pi_p \end{pmatrix}$ of $1, 2, \cdots, p$ we may define a tensor

$$A^{(i)}_{(j)} = T^{(i\pi)}_{(j)} \equiv T^{i_{\pi_1} i_{\pi_2} \cdots i_{\pi_p}}_{j_1 j_2 \cdots j_p},$$

an *isomer* of $T^{(i)}_{(j)}$. If the indices $(i) = i_1 i_2 \cdots i_p$ in (10) are subjected to the permutation π, the right-hand side of the equation appears with the tensor $A^{(i)}_{(j)}$ replacing $T^{(i)}_{(j)}$. As to the left-hand side, the first term becomes

$$F_\beta{}^{i_{\pi_1}} \cdot T^{\beta i_{\pi_2} \cdots i_{\pi_p}}_{(j)} = F_\beta{}^{i_{\pi_1}} \cdot A^{i_1 i_2 \cdots \beta \cdots}_{(j)},$$

where in $A^{(i)}_{(j)}$ the index i_{π_1} has been replaced by β. A similar remark applies to the other terms. Hence if the terms of the left-hand side are rearranged so that $F_\beta{}^{i_1}, F_\beta{}^{i_2}, \cdots, F_\beta{}^{i_p}$ are written in order, the new left-hand side differs from that of (10) only in that $T^{(i)}_{(j)}$ has been replaced by $A^{(i)}_{(j)}$. In other words, if $F_j{}^i$ satisfies the equation (10), it must satisfy each of the tensor equations obtained from (10) by substituting for $T^{(i)}_{(j)}$ any of its isomers $T^{(i\pi)}_{(j)}$ or, more generally, any linear sum of such isomers. Now $T^{(i)}_{(j)}$ may be decomposed, with respect to its p contravariant indices (i), into a sum of tensors

(11) $$^{[\alpha]}T^{(i)}_{(j)} = {}_{[\alpha]}I^{(i)}_{(l)} \cdot T^{(l)}_{(j)},$$

where

(12) $${}_{[\alpha]}I^{(i)}_{(l)} = {}_{[\alpha]}I^{i_1 i_2 \cdots i_p}_{l_1 l_2 \cdots l_p}$$

is the normalized (numerical) immanant tensor corresponding to the Young tableau $[\alpha]$.[2] The tensor equation (10) may be replaced by an equivalent set of tensor equations obtained by substituting for $T^{(i)}_{(j)}$ in turn each of the tensors (11). Each tensor (11) may be decomposed further, with respect to the p indices (i), into a sum of f_α tensors corresponding to the f_α standard diagrams of the tableau $[\alpha]$, and the tensor equation corresponding to $^{[\alpha]}T^{(i)}_{(j)}$ may be replaced by an equivalent set of f_α tensor equations.

Similar remarks may be made in regard to tensor decomposition relative

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[2] For the definitions and properties of the immanant tensors, see the paper by T. L. Wade [2], which also contains extensive references to the literature on decomposition of tensors. For decomposition with respect to the standard diagrams, see [3], Part II, section 5.

to the q covariant indices $(j) = j_1 j_2 \cdots j_q$. For example we may obtain from $T^{(i)}_{(j)}$ a tensor

$$(13) \qquad {}^{[\alpha]}_{[\beta]}T^{(i)}_{(j)} = {}_{[\alpha]}I^{(i)}_{(l)} \cdot {}_{[\beta]}I^{(m)}_{(i)} \cdot T^{(l)}_{(m)}$$

of Young type $[\alpha]$ on its p superscripts and Young type $[\beta]$ on its q subscripts; and this tensor may further be decomposed into a sum of $f_\alpha \cdot f_\beta$ tensors. To sum up, *we may symmetrize independently the upper and lower indices of* $T^{(i)}_{(j)}$ *and replace* (10) *by a set of tensor equations obtained by replacing* $T^{(i)}_{(j)}$ *in turn by each of its "symmetrized parts."* This result would appear to be the equivalent of the fact that the invariants of a tensor $T^{(i)}_{(j)}$ are the simultaneous invariants of its symmetrized parts.

When pq is $\neq 0$ the symmetrized parts of $T^{(i)}_{(j)}$ obtained as above are not primitive.[3] Although we shall not be concerned here with further decomposition, nevertheless it seems worthwhile to indicate how the primitive parts may be obtained. In addition to the above two processes of symmetrization, we introduce the following "contraction" process: Equate any two indices i_λ, j_μ of $T^{(i)}_{(j)}$ and contract; then, to restore the lost indices, multiply the contracted tensor by $\delta_{j_\mu}{}^{i_\lambda}$. By application of the same process to $\xi^{(i)\beta}_{(j)\alpha}F_\beta{}^\alpha$ it is easily verified that in (10) the tensor $T^{(i)}_{(j)}$ may be replaced by the tensor so derived. The three processes are sufficient to break up $T^{(i)}_{(j)}$ into its primitive parts. Where $\epsilon_{i_1 i_2 \ldots i_n}$ and $\epsilon^{i_1 i_2 \cdots i_n}$ are the well-known relative numerical tensors we may arrive at the same goal by a different path via the equations

$$(14) \qquad U_{(j)(k_1)(k_2)\ldots(k_p)} = T^{(l)}_{(j)} \cdot \epsilon_{l_1(k_1)} \cdot \epsilon_{l_2(k_2)} \ldots \epsilon_{l_p(k_p)}$$

and

$$(15) \qquad T^{(i)}_{(j)} = \frac{1}{(n-1!)^p} \cdot \epsilon^{i_1(k_1)} \cdots \epsilon^{i_p(k_p)} \cdot U_{(j)(k_1)\ldots(k_p)};$$

we obtain the primitive parts of U by symmetrization and use (15) to obtain the primitive parts of T.

2. Values of N for tensors of various types.[4] Theorem I makes it desirable to determine the numerical invariants N and Q for tensors of any type; in this section we deal with N. Let $C^{(i)}_{(l)}$, $D^{(m)}_{(j)}$, where

$$(16) \qquad C^{(i)}_{(l)} = C^{i_1 i_2 \cdots i_p}_{l_1 l_2 \cdots l_p}, \qquad D^{(m)}_{(j)} = D^{m_1 m_2 \cdots m_q}_{j_1 j_2 \cdots j_q},$$

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[3] Cf. [4], pp. 131-132.

[4] This section is based almost entirely upon [3], Part II, especially upon sections 4 and 5.

be *prepared* idempotent numerical tensors corresponding, respectively, to one of the f_a standard diagrams of a Young tableau form $[\alpha] = [\alpha_1, \alpha_2, \cdots, \alpha_p]$ and to one of the f_β standard diagrams of a Young tableau form $[\beta] = [\beta_1, \beta_2, \cdots, \beta_q]$ so that [5]

(17) $$u \equiv C^{(i)}_{(i)} = r_a/f_a, \qquad v \equiv D^{(j)}_{(j)} = r_\beta/f_\beta.$$

Let the tensor

(18) $$A^{(i)}_{(j)} = A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$$

range over all tensors which satisfy

(19) $$C^{(i)}_{(l)} A^{(l)}_{(m)} D^{(m)}_{(j)} = A^{(i)}_{(j)}.$$

In this case we have

LEMMA I. *The number of functionally independent components of* $A^{(i)}_{(j)}$ *is* $u \cdot v = r_a r_\beta / f_a f_\beta$.

Proof. The number $u = C^{(i)}_{(i)}$ is shown in [3] (Part II, section 2) to be the greatest integer r for which the tensor

(20) $$C^{(i_1)\cdots(i_r)}_{(l_1)\cdots(l_r)} = \begin{vmatrix} C^{(i_1)}_{(l_1)} & \cdot & \cdot & \cdot & C^{(i_1)}_{(l_r)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ C^{(i_r)}_{(l_1)} & \cdot & \cdot & \cdot & C^{(i_r)}_{(l_r)} \end{vmatrix}$$

does not vanish. Equivalently, therefore, it is equal to the greatest number of independent choices of the superscripts (i) or subscripts (l) of $C^{(i)}_{(l)}$. A similar interpretation holds for $v = D^{(j)}_{(j)}$, and the Lemma follows by (19) and (17).[6] Again:

THEOREM II. *The number of independent components of a tensor* ${}_{[\alpha]}^{[\alpha]}T^{(i)}_{(j)}$ *of Young type* $[\alpha]$ *on the* p *contravariant indices* (i) *and of Young type* $[\beta]$ *on the* q *covariant indices* (j) *is* $r_a \cdot r_\beta$.

Proof. The tensor may be decomposed into a sum of $f_a \cdot f_\beta$ tensors such as $A^{(i)}_{(j)}$; hence the theorem follows from Lemma I.

As a consequence of the theorem a completely arbitrary tensor $T^{(i)}_{(j)}$ ought to have independent components numbering

[5] The number r_a is the trace $I^{(i)}_{(i)}$ of the numerical idempotent tensor ${}_{[a]}I$ of symmetry type $[a]$.

[6] For a neat proof of a comparable result, see [4a]. On p. 471, line 1 of that paper, $T_{(j)}$ should be ${}_{[a]}T_{(j)}$.

$$\sum_{[\alpha]} \sum_{[\beta]} r_\alpha r_\beta = \sum_{[\alpha]} r_\alpha \cdot \sum_{[\beta]} r_\beta,$$

where $[\alpha]$ is summed over all partitions of p and $[\beta]$ over all partitions of q. In virtue of the known identities $\Sigma r_\alpha = n^p$, $\Sigma r_\beta = n^q$ this gives the correct result: $n^p \cdot n^q = n^{p+q}$.

Of particular interest, in the case of $q = p$, is the algebra of bisymmetric tensors

$$(21) \qquad B^{(i)}_{(j)} = B^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_p}, \qquad B^{(i\pi)}_{(j\pi)} = B^{(i)}_{(j)} \text{ for each } \pi,$$

which we consider next. Now $I^{(i)}_{(m)} B^{(m)}_{(j)} = B^{(i)}_{(m)} \cdot I^{(m)}_{(j)}$ for every numerical tensor $I^{(i)}_{(j)}$; accordingly ${}^{[\alpha]}_{[\beta]}B^{(i)}_{(j)} = 0$ unless $[\alpha] = [\beta]$. Thus we may write

$$(22) \qquad B^{(i)}_{(j)} = \sum_{[\alpha]} {}_{[\alpha]}B^{(i)}_{(j)}.$$

Let $C^{(i)}_{(j)}$ be the numerical idempotent tensor associated with the natural diagram of the tableau form $[\alpha]$, and let ${}_{[\alpha]}I^{(i)}_{(j)}$ be the normalized immanant tensor of type $[\alpha]$. Then, corresponding to the equations given in [4], p. 126, we have in our notation

$$(23) \qquad {}_{[\alpha]}I^{(i)}_{(j)} = (f_\alpha/p!) \sum_{\pi} C^{(i\pi)}_{(j\pi)}$$

where π is summed over all permutations of $1, 2, \cdots, p$; and

$$(24) \qquad {}_{[\alpha]}I^{(i)}_{(l)} C^{(l\pi)}_{(j\pi)} = C^{(i\pi)}_{(j\pi)}, \text{ all } \pi.$$

A bisymmetric tensor ${}_{[\alpha]}B^{(i)}_{(j)}$ uniquely determines a tensor

$$(25) \qquad N^{(i)}_{(j)} = C^{(i)}_{(l)}\, {}_{[\alpha]}B^{(l)}_{(j)}$$

which clearly satisfies

$$(26) \qquad C^{(i)}_{(l)} N^{(l)}_{(m)} C^{(m)}_{(j)} = N^{(i)}_{(j)}$$

and thus may be said to be associated with the natural diagram. Conversely $N^{(i)}_{(j)}$, when defined by (25), uniquely determines ${}_{[\alpha]}B^{(i)}_{(j)}$. For each permutation π

$$N^{(i\pi)}_{(j\pi)} = C^{(i\pi)}_{(l)}\, {}_{[\alpha]}B^{(l)}_{(j\pi)} = C^{(i\pi)}_{(l\pi)}\, {}_{[\alpha]}B^{(l\pi)}_{(j\pi)} = C^{(i\pi)}_{(l\pi)}\, {}_{[\alpha]}B^{(l)}_{(j)},$$

and hence

$$(27) \quad (f_\alpha/p!) \cdot \sum_{\pi} N^{(i\pi)}_{(j\pi)} = (f_\alpha/p!) \sum_{\pi} C^{(i\pi)}_{(l\pi)}\, {}_{[\alpha]}B^{(l)}_{(j)} = {}_{[\alpha]}I^{(i)}_{(l)}\, {}_{[\alpha]}B^{(l)}_{(j)} = {}_{[\alpha]}B^{(i)}_{(j)}.$$

We are now ready to prove

THEOREM III. *The number of independent components of a bisymmetric tensor* ${}_{[a]}B^{(i)}_{(j)}$ *of type* $[\alpha]$ *is* $(r_a/f_a)^2$.

By virtue of Lemma I we need only show that if $N^{(i)}_{(j)}$ is any tensor satisfying (24) the tensor

$$P^{(i)}_{(j)} = (f_a/p!) \sum_{\pi} N^{(i\pi)}_{(j\pi)} \tag{28}$$

is bisymmetric of type $[\alpha]$. Clearly $P^{(i)}_{(j)}$ is bisymmetric, from its definition. On the other hand, by (26), (24),

$${}_{[a]}I^{(i)}_{(l)} N^{(l\pi)}_{(j\pi)} = {}_{[a]}I^{(i)}_{(l)} C^{(l\pi)}_{(m\pi)} N^{(m\pi)}_{(j\pi)} = C^{(i\pi)}_{(m\pi)} N^{(m\pi)}_{(j\pi)} = N^{(i\pi)}_{(j\pi)};$$

whence

$${}_{[a]}I^{(i)}_{(l)} P^{(l)}_{(j)} = P^{(i)}_{(j)}. \tag{29}$$

COROLLARY. *The number of independent components of an arbitrary bisymmetric tensor is*[7]

$$\binom{n^2+p-1}{p} = \sum_{[a]} \left(\frac{r_a}{f_a}\right)^2. \tag{30}$$

Proof. With respect to the right-hand side of (30), the corollary is established by (22) and Theorem III. As to the left-hand side, the number of independent components of an arbitrary bisymmetric tensor (21) is clearly equal to the number of choices, repetitions allowed, of p index-pairs $\begin{smallmatrix} i_1, & i_2, & \ldots, & i_p \\ j_1 & j_2 & & j_p \end{smallmatrix}$ from the n^2 pairs $\begin{smallmatrix} 1, & 2, & \ldots, & n \\ 1 & 2 & & n \end{smallmatrix}$: that is, to $\binom{n^2+p-1}{p}$.

In view of [6] and [3] the values obtained for N in this section may be regarded as explicit. We might elaborate the problem of finding N by considering tensors subject to invariantive restrictions of other types;[8] we limit ourselves however to a single example. The number of independent components of a *coördinate tensor* $T_{i_1 i_2 \ldots i_p}$ is $N = p(n-p)+1$, being one larger than the result given by Bertini [7] for the number of independent homogeneous coördinates of a $(p-1)$-dimensional linear subspace of a linear $(n-1)$-space.

3. Values of Q. The determination of Q appears to be more difficult than the problem just considered, and the results of this section are less complete than those of 2. For a given tensor $T^{(i)}_{(j)}$, let $L(T)$ denote the number

[7] Compare [6], pp. 124, 129, formulas (4.37) and (4.41).

[8] In this connection note [4], p. 242, Theorem (8.1B).

of linearly independent tensors $F_j{}^i$ which satisfy the tensor equation (10), and define

$$(31) \qquad Q(T) = n^2 - L(T).$$

If $T^{(i)}_{(j)}$ ranges over a set of tensors $\mathfrak{S}$, as for example over all tensors skew-symmetric on the p indices (i), we define the generic number $Q = Q(\mathfrak{S})$ $(L = L(\mathfrak{S}))$ to be the greatest of the integers $Q(T)$ (the least of the integers $L(T)$). Clearly

$$(32) \qquad Q = n^2 - L; \quad 0 \leqq Q(T) \leqq Q \leqq n^2; \quad 0 \leqq L \leqq L(T) \leqq n^2.$$

The following tentative procedure has proved useful. If Q^* is an upper bound for Q we attempt to choose a $T^{(i)}_{(j)} \subset \mathfrak{S}$ for which $Q(T) = Q^*$; when this attempt is successful, (32) implies $Q = Q^*$; when it is unsuccessful we attempt to replace Q^* by a smaller upper bound. As an example of a Q^* we cite

LEMMA II. *If* $p = q > 0$, *then* $Q \leqq n^2 - 1$.

In fact the tensor $F_j{}^i = \delta_j{}^i$ is a solution of (10), for $T^{(i)}_{(j)} \neq$ the null tensor, when and only when $q = p$. As a further example, suppose that we have been able to construct A independent absolute invariants of T—by the method of Cramlet [8], let us say; then (Theorem I) $N - Q \geqq A$, whence $Q \leqq Q^* = N - A$.

For a vector T_i, or, more generally, for a coördinate tensor

$$(33) \qquad T_{i_1 i_2 \ldots i_p} = \begin{vmatrix} x_{i_1} & y_{i_1} & \cdot & \cdot & \cdot & w_{i_1} \\ x_{i_2} & y_{i_2} & \cdot & \cdot & \cdot & w_{i_2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{i_p} & y_{i_p} & \cdot & \cdot & \cdot & w_{i_p} \end{vmatrix}, \qquad (p \leqq n),$$

we readily deduce $N = Q$ from the fact that such a tensor has no non-constant invariants; for clearly by choice of coördinates we may arrange that

$$(33a) \qquad T_{i_1 i_2 \ldots i_p} = \begin{vmatrix} \delta_{i_1}{}^1 & \delta_{i_1}{}^2 & \cdot & \cdot & \cdot & \delta_{i_1}{}^p \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \delta_{i_p}{}^1 & \delta_{i_p}{}^2 & \cdot & \cdot & \cdot & \delta_{i_p}{}^p \end{vmatrix}.$$

It is interesting to calculate Q and hence N in this case by means of (10) and (33a).

As the tensors $T_j{}^i$ and T_{ij} prove exceptions to a theorem which we wish to announce below, we shall consider these briefly.

$T_j{}^i$. For this tensor, with $T=(T_j{}^i)$, $F=(F_j{}^i)$, equations (10) may be written in matrix notation as

$$FT = TF. \tag{34}$$

The tensor has n absolute invariants, the n coefficients of the characteristic equation $|F-\lambda\delta_j{}^i|=0$, which are seen to be independent if we take

$$T = \begin{pmatrix} \alpha_1 & & & & \\ & \alpha_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \alpha_p \end{pmatrix}, \tag{35}$$

where the $n+1$ numbers $0, \alpha_1, \alpha_2, \cdots, \alpha_p$ are all distinct. The same expression for T, used in (34), forces F to be a diagonal matrix, so that $L(T)=n$. But $L=n^2-Q=N-Q \geqq n$, whence $L=L(T)=n$, $Q=n^2-n$. Thus, as is well known, $N-Q=n$.

T_{ij}. If we define

$$\begin{aligned} S_{ij} &= \tfrac{1}{2}(T_{ij}+T_{ji}), \qquad S=(S_{ij}), \\ A_{ij} &= \tfrac{1}{2}(T_{ij}-T_{ji}), \qquad A=(A_{ij}), \end{aligned} \tag{36}$$

then (10) is equivalent to the equations

$$SF + F'S = 0, \qquad AF + F'A = 0, \tag{37}$$

where F' is the transpose of $F=(F_j{}^i)$. We put aside the cases where T_{ij} is symmetric ($N=Q=n(n+1)/2$) and where T_{ij} is skew-symmetric ($N=Q=n(n-1)/2$), and assume it to be arbitrary. The polynomial

$$f(\lambda) = |S_{ij}+\lambda A_{ij}| = |S+\lambda A|, \tag{38}$$

being the determinant of a covariant tensor, is a relative invariant; moreover, since $f(\lambda)=|S'+\lambda A'|=|S-\lambda A|=f(-\lambda)$, it follows that $f(\lambda)=g(\lambda^2)$ is a polynomial in λ^2. If $m=[n/2]$ (the greatest integer in $n/2$), $g(\lambda^2)$ has $m+1$ coefficients, relative invariants of equal weight, whose ratios give rise to m absolute invariants. That these are independent may be verified by calculating $g(\lambda^2)$ when

$$S = S_1 = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}, \qquad A = A_1 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad (\text{for } n=2m) \tag{39}$$

or

$$(39a)\qquad S=\begin{pmatrix} S_1 & 0\\ 0 & 1\end{pmatrix},\qquad A=\begin{pmatrix} A_1 & 0\\ 0 & 0\end{pmatrix}\quad (\text{for } n=2m+1).$$

Here $I=I_m$ is the unit matrix and $D=D_m$ is an arbitrary diagonal matrix. Using these choices of S, A in (37) we may deduce $Q(T)=n^2-m$; whence, by the same type of reasoning as before, $Q=n^2-[n/2]$, $N-Q=[n/2]$. When the $[n/2]$ roots of $g(\mu)=f(\mu^{1/2})=0$ are distinct, finite and $\neq 0$, and lie in the field under consideration, (39) (or (39a)) gives a canonical form for the pair S, A and hence for the tensor T_{ij}. Whether or not the roots are distinct, the corresponding canonical forms may be deduced in the usual manner by choice of a "characteristic" coördinate system.

THEOREM IV. *Let the part ${}^{[p]}_{[q]}T^{(i)}_{(j)}$ of the tensor $T^{(i)}_{(j)}$ range over all tensors symmetric in the indices (i) and in the indices (j). Except for the cases, treated above, in which $p+q \leqq 2$, we have $Q=n^2$ if $p\neq q$ and $Q=n^2-1$ if $p=q$.*

Since the exceptional cases have been considered, we take $p+q>2$. Without real loss of generality we may assume $q\geqq p$ and hence $q>1$. The cases $p=1$, $q=2$ and $p=q=2$ require special treatment. In the former case we may set

$$T^i{}_{j_1j_2}=\begin{cases}1, & \text{if } i=j_1=j_2,\\ 0, & \text{otherwise,}\end{cases}$$

and prove that $F_j{}^i=0$. In the latter we may set

$$T^{i_1i_2}_{j_1j_2}=\begin{cases}c_j{}^i, & \text{if } i_1=i_2=i,\ j_1=j_2=j,\\ 0, & \text{otherwise,}\end{cases}$$

where $c_j{}^i$ is $\neq 0$ if and only if $i\neq j$; in this case it turns out that $F=f\cdot\delta_j{}^i$, an arbitrary multiple of the Kronecker delta. In each case the conclusions of Theorem IV follow immediately. We now assume $q>2$. Instead of completing the proof of IV we shall prove a more general theorem which includes it in this case.

THEOREM V. *Assume $q\geqq \max(p,3)$. Let $[\alpha]$ be any partition of p for which $r_\alpha>0$, and let the part ${}^{[\alpha]}_{[q]}T^{(i)}_{(j)}$ of the tensor $T^{(i)}_{(j)}$ range over all tensors symmetric in the q indices (j) and of type $[\alpha]$ in the p indices (i). Then $Q=n^2$, n^2-1 according as $q>p$, $q=p$.*

Proof of V. For $p=0$ or 1 it is convenient to define $r_\alpha=1$, in order to include these cases in the theorem. Noting that in particular $F_j{}^i$ must

satisfy (10) when ${}^{[a]}_{[q]}T'^{(i)}_{(j)}$ is substituted for $T^{(i)}_{(j)}$, we make the following specialization:

$$T^{(i)}_{(j)} = {}^{[a]}_{[q]}T^{(i)}_{(j)} = \begin{cases} T_j^{(i)} & \text{if } j_1 = j_2 = \cdots = j_q = j, \\ 0 & \text{otherwise,} \end{cases} \tag{40}$$

where the set of quantities $T_j^{(i)}$ are so chosen that, independently of $j = 1, 2, \cdots, n$, there exists a fixed set of indices (i) for which $T_j^{(i)} \neq 0$. Using (40) in (10), we first set $j_2 = \cdots = j_q = j \neq j_1$, whence

$$0 = F_{j_1}{}^{j} T_j{}^{(i)} \quad \text{(not summed for } j\text{)}, \tag{41}$$

or $F_j{}^i = 0$ for $i \neq j$. Next, using this fact, we set $j_1 = j_2 = \cdots = j_q = j$ in (10) and obtain, after cancellation of $T_j{}^{(i)} \neq 0$,

$$F_{i_1}{}^{i_1} + \cdots + F_{i_p}{}^{i_p} = qF_j{}^j \quad \text{(no summations)}. \tag{42}$$

Since the quantity on the left of (42) is independent of j, it is evident that $F_1{}^1 = F_2{}^2 = \cdots = F_n{}^n$; thus, for $i_1 = \cdots = i_p = j = 1$ in (42), there follows

$$pF_1{}^1 = qF_1{}^1. \tag{43}$$

From (43), if $p \neq q$, it is seen that $F_j{}^i = 0$, all i, j; and hence $n^2 - Q(T) = 0$, $Q = Q(T) = n^2$. But if $p = q$ there is no restriction on $F_1{}^1$, so $n^2 - Q(T) = 1$, $Q(T) = n^2 - 1$; and by Lemma II we have $Q = n^2 - 1$.

We list a few consequences of Theorems I to V:

(1) *A symmetric tensor* $T_{i_1 i_2 \ldots i_p}$ *has, for* $p > 2$, $\binom{n+p-1}{p} - n^2$ *functionally independent absolute invariants, and none for* $p = 1, 2$. [9].

(2) *For* $p + q > 2$, *an arbitrary tensor* $T^{(i)}_{(j)} = T^{j_1 j_2 \cdots j_q}_{i_1 i_2 \cdots i_p}$ *has* $n^{p+q} - n^2$ *or* $n^{2p} - n^2 - 1$ *functionally independent absolute invariants, according as* $q \neq p$, $q = p$. *On the other hand the arbitrary tensors* $T_j{}^i$ *and* T_{ij} *have respectively* n *and* $[n/2]$ *functionally independent absolute invariants.*

(3) *A bisymmetric tensor* $B^{(i)}_{(j)} = B^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_p}$, $(p > 1)$, *has* $\binom{n^2+p-1}{p} - n^2 + 1$ *functionally independent absolute invariants.*

(4) *A tensor* $T^{(i)}_{(j)} = T^{i_1 i_2}_{j_1 j_2 j_3}$ *symmetric in* $i_1 i_2$ *and of type* $[2, 1]$ *in* $j_1 j_2 j_3$ *has* $r_{[2]} \cdot r_{[2,1]} - n^2 = \binom{n+1}{2} \cdot 4 \binom{n+1}{3} - n^2$ *functionally independent absolute invariants.*

However the above theorems give no information, for example, as to the number of invariants of a skew-symmetric tensor A_{ijk}.

4. Adaptation of the theory to relative tensors. In many cases the use of relative tensors is not so much a necessity [9] as a matter of convenience. For example, equation (14) allows us to replace a mixed tensor by a relative covariant tensor. Again, for $n = 5$, instead of considering a 3-indexed skew-symmetric tensor $S_{i_1 i_2 i_3}$ we may with advantage study the (dual) 2-indexed tensor $R^{i_1 i_2} = (1/3\,!)\epsilon^{i_1 i_2 i_3 i_4 i_5} S_{i_3 i_4 i_5}$. Thus it seems worthwhile to adapt the preceding theory to relative tensors.

Let $T^{(i)}_{(j)}$ be a relative tensor of weight $w \neq 0$, so that the law of transformation is

$$\bar{\bar{T}}{}^{(i)}_{(j)} = a^{-w} \cdot \bar{T}{}^{(i)}_{(j)}, \qquad a = | a_j{}^i |, \tag{44}$$

where $\bar{T}{}^{(i)}_{(j)}$ is defined by (1). The theorems of **2** and **3** are valid without change. In **1**, (7) must be replaced by

$$\partial \bar{\bar{T}}{}^{(i)}_{(j)} / \partial a_\beta{}^\alpha = a^{-w} \cdot (\partial \bar{T}{}^{(i)}_{(j)} / \partial a_\beta{}^\alpha) - w \cdot a^{-w} \cdot \bar{T}{}^{(i)}_{(j)} \cdot \alpha_\beta{}^\alpha. \tag{45}$$

Hence if

$$\eta^{(i)\beta}_{(j)\alpha} = \partial \bar{\bar{T}}{}^{(i)}_{(j)} / \partial a_\beta{}^\alpha]_{a=a_0}, \tag{46}$$

we have

$$\eta^{(i)\beta}_{(j)\alpha} = \xi^{(i)\beta}_{(j)\alpha} - w \cdot T^{(i)}_{(j)} \cdot \delta_\alpha{}^\beta, \tag{47}$$

where $\xi^{(i)\beta}_{(j)\alpha}$ is defined by (9). To adapt Theorems I, I′ to the present situation we need merely replace ξ by η and Q by Q'. Here $n^2 - Q$, $n^2 - Q'$ are the generic numbers of linearly independent tensors $F_j{}^i$ which satisfy respectively the tensor equation

$$\xi^{(i)\beta}_{(j)\alpha} \cdot F_\beta{}^\alpha = 0 \tag{48}$$

and the tensor equation

$$\eta^{(i)\beta}_{(j)\alpha} \cdot F_\beta{}^\alpha \equiv \xi^{(i)\beta}_{(j)\alpha} \cdot F_\beta{}^\alpha - w T^{(i)}_{(j)} \cdot F_\alpha{}^\alpha = 0. \tag{49}$$

In order that $F_j{}^i$ should satisfy both of (48), (49) it is necessary that its trace $F_\alpha{}^\alpha$ should be zero; conversely a solution with zero trace of one of (48), (49) is a solution of both. If $F_j{}^i$ is a solution of (48) with non-zero trace $f = F_\alpha{}^\alpha \neq 0$, and if $G_j{}^i$ is any other solution, the solution

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[9] [10].

$$G_j{}^i - (1/f) \cdot G_a{}^a \cdot F_j{}^i$$

has zero trace; thus a set of $n^2 - Q$ linearly independent solutions of (48) may be chosen such that at most one has non-zero trace. Since a similar remark is true for (49) we have that *Q' and Q can differ at most by unity.* More precise results are embodied in

THEOREM VII. *Let $T^{(i)}_{(j)} = T^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ be a relative tensor of weight $w \neq 0$. Let N, Q have the same significance as in Theorem I′, and let A denote the number of functionally independent absolute invariants of T. Then*: (1) *if $p - q \neq 0$, nw, we have $A = N - Q$*; (2) *if $p = q$ but $T^{(i)}_{(i)} \neq 0$, we have $A = N - Q - 1$.*

Proof. (1) Consider the linear transformation

$$F_j{}^i = G_j{}^i - [w/(p - q)] \cdot G_a{}^a \cdot \delta_j{}^i \tag{50}$$

of $G_j{}^i$ into $F_j{}^i$. After contraction and rearrangement it may be verified that

$$\frac{w}{p - q - nw} \cdot F_a{}^a = \frac{w}{p - q} \cdot G_a{}^a; \tag{51}$$

hence (50) is non-singular, with inverse transformation

$$G_j{}^i = F_j{}^i + \frac{w}{p - q - nw} \cdot F_a{}^a \cdot \delta_j{}^i. \tag{52}$$

From either (50) or (52), by use of the identity

$$\xi^{(i)a}_{(j)a} = (p - q) T^{(i)}_{(j)}, \tag{53}$$

(cf. (9)), we find

$$\xi^{(i)\beta}_{(j)a} F_\beta{}^a = \eta^{(i)\beta}_{(j)a} G_\beta{}^a. \tag{54}$$

Hence (50) or (51) sets up a one-to-one correspondence between the solutions $G_j{}^i$ of (49) and $F_j{}^i$ of (48). It follows that $Q' = Q$.

(2) When $p = q$, we have (from (9) and (47), by contraction) the identity

$$\eta^{(i)\beta}_{(i)a} F_\beta{}^a = - w T^{(i)}_{(i)} \cdot F_a{}^a; \tag{55}$$

hence any solution of (49) has zero trace. On the other hand $F_j{}^i = \delta_j{}^i$ is a solution with non-zero trace of (48). Thus $n^2 - Q = (n^2 - Q') + 1$ or $N - Q' = N - Q - 1$.

The relative covariant tensor U defined by (14) in **1** has weight $w' = -p$,

and $q' = q + (n - 1)p$ covariant indices. Since $p' - q' - nw' = p - q$, the preceding theorem applies to U if $p \neq q$.

University of Wisconsin.

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CONTINUA OF FINITE LINEAR MEASURE, II.*

By Samuel Eilenberg.

1. The convexification theorem. In a previous paper [1] conditions were established for a continuum X in order that there exist a metrization ρ such that the linear measure $L^1(X,\rho)$ be finite. We shall show here that every such metric ρ can be expanded to a convex metric without increasing the linear measure. This convex metric is unique and has certain maximal properties which will be explained in the sequel.

Theorem I. *Let X be a continuum and ρ a metric for X such that $L^1(X,\rho) < \infty$. There is a metric ρ^* for X such that:* [2]

(1) $\rho^* \geqq \rho,$

(2) *ρ^* is convex,*[3]

(3) $L^1(X,\rho^*) = L^1(X,\rho).$

The proof of Theorem I will be given in **4**.

2. L^1-equivalent metrics. Two metrics ρ_1 and ρ_2 for a continuum X will be called *L^1-equivalent* if $L^1(A,\rho_1) = L^1(A,\rho_2)$ for every subset A of X.

Lemma 1. *If $\rho_2 \geqq \rho_1$ and $L^1(X,\rho_1) = L^1(X,\rho_2) < \infty$ then ρ_1 and ρ_2 are L^1-equivalent.*

Proof. Let $A \subset X$. Since the linear measure is a regular Carathéodory measure there are [4] G_δ sets G_1 and G_2 such that $A \subset G_i$ and $L^1(A,\rho_i) = L^1(G_i,\rho_i)$; $(i = 1, 2)$. Taking $G = G_1G_2$ we have then $A \subset G$ and $L^1(A,\rho_i) = L^1(G,\rho_i)$, $(i = 1, 2)$. Since G is measurable we also have $L^1(X,\rho_i) = L^1(G,\rho_i) + L^1(X - G,\rho_i)$; $(i = 1, 2)$; and therefore

* Received August 18, 1943. Presented to the American Mathematical Society, November 23, 1940.

[1] S. Eilenberg and O. G. Harrold, "Continua of finite linear measure I," *American Journal of Mathematics*, vol. 65 (1943), pp. 137-146. We shall refer to this paper as I. The reader will find there the definition of linear measure and a bibliography.

[2] Since ρ^* is a metric for X, it goes without saying that ρ and ρ^* are topologically equivalent.

[3] This means that given $x_1, x_2 \in X$ there is an $x \in X$ such that $\rho^*(x_1,x) = \rho^*(x,x_2) = \frac{1}{2}\rho^*(x_1,x_2)$.

[4] S. Saks, *Theory of the Integral*, Warsaw (1937), p. 53.

$$L^1(G,\rho_1)+L^1(X-G,\rho_1)=L^1(G,\rho_2)+L^1(X-G,\rho_2).$$

But since $\rho_2 \geqq \rho_1$ we also have

$$L^1(G,\rho_2)\geqq L^1(G,\rho_1), \qquad L^1(X-G,\rho_2)\geqq L^1(X-G,\rho_1);$$

hence $L^1(G,\rho_1)=L^1(G,\rho_2)$ and consequently $L^1(A,\rho_1)=L^1(A,\rho_2)$.

COROLLARY. *The metrics ρ and ρ^* of Theorem* I *are L^1-equivalent.*

3. L^1-equivalence and convexity.

LEMMA 2. *If ρ_1 and ρ_2 are L^1-equivalent and ρ_2 is convex then $\rho_2 \geqq \rho_1$.*

Proof. Let $x_1, x_2 \in X$ and let A be an arc geodesic relative to ρ_2 [5] joining x_1 and x_2. We have

$$\rho_2(x_1,x_2)=L^1(A,\rho_2)=L^1(A,\rho_1)\geqq \rho_1(x_1,x_2);$$

hence $\rho_2 \geqq \rho_1$.

COROLLARY. *Two convex L^1-equivalent metrics are identical.*

COROLLARY. *The metric ρ^* of Theorem* I *is unique.*

From Theorem I and the last three corollaries we obtain

THEOREM II. *Let X be a continuum, ρ a metric for X such that $L^1(X,\rho) < \infty$ and R the class of metrics L^1-equivalent with ρ. There is in R a maximal element which is convex and is the only convex element of R.*

4. Proof of Theorem I. Since $L^1(X,\rho)<\infty$, X is a locally connected continuum; [6] *a fortiori* X is then arcwise connected. Define

$$\rho^*(x_1,x_2)=\text{g. l. b. } L^1(A,\rho)$$

where A varies over all arcs joining x_1 and x_2. It is clear that ρ^* satisfies all the formal postulates of a metric and also that $\rho^* \geqq \rho$ since $L^1(A,\rho) \geqq \rho(x_1,x_2)$. We shall prove that ρ^* is a metric for the continuum X. Consider a sequence $x_0, x_1, \cdots$ in X. If $\lim_{n\to\infty}\rho^*(x_n,x_0)=0$ then since $\rho^*\geqq\rho$,

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[5] Meaning an arc $A=\overline{x_1x_2}$ such that $\rho_2(x_1,x)+\rho_2(x,x_2)=\rho_2(x_1,x_2)$ for all $x \in A$. The existence of a geodesic arc for a convex metric is a well known theorem, provided X is complete. In our case X is compact.

[6] I, p. 140.

also $\lim_{n\to\infty} \rho(x_n, x_0) = 0$ and therefore $\lim_{n\to\infty} x_n = x_0$. Conversely if $\lim_{n\to\infty} x_n = x_0$, then, since X is locally connected, there is a sequence of arcs $A_n = \overline{x_n, x_0}$ such that $\lim_{n\to\infty} A_n = x_0$. Since $L^1(X, \rho) < \infty$ and the linear measure is a Carathéodory measure it follows [7] that $\lim_{n\to\infty} L^1(A, \rho) = 0$ and therefore $\lim \rho^*(x_n, x_0) = 0$ since $L^1(A_n, \rho) \geqq \rho^*(x_n, x_0)$. We now prove that ρ^* is a convex metric. Let $x_1, x_2 \epsilon X$. Given $\epsilon > 0$ there is an arc $A = \overline{x_1 x_2}$ such that

$$\epsilon + \rho^*(x_1, x_2) > L^1(A, \rho).$$

Let $x \epsilon A$ and let A_x be the subarc $\overline{x_1 x}$ of A. As x varies from x_1 to x_2, the linear measure $L^1(A_x, \rho)$ varies continuously from 0 to $L^1(A, \rho)$ and therefore there is a point x_0 which divides A into two arcs $A_1 = \overline{x_1 x_0}$ and $A_2 = \overline{x_0 x_2}$ such that

$$L^1(A_1, \rho) = L^1(A_2, \rho) = \tfrac{1}{2} L^1(A, \rho).$$

Since $L^1(A_1, \rho) \geqq \rho^*(x_1, x_0)$ and $L^1(A_2, \rho) \geqq \rho^*(x_0, x_2)$ we see that

$$\rho^*(x_1, x_0) < \tfrac{1}{2}\rho^*(x_1, x_2) + \epsilon/2, \qquad \rho^*(x_0, x_2) < \tfrac{1}{2}\rho^*(x_1, x_2) + \epsilon/2.$$

Since X is compact and such a point x_0 exists for every $\epsilon > 0$ it follows readily that ρ^* is convex.

Since $\rho^* \geqq \rho$ we clearly have $L^1(X, \rho^*) \geqq L^1(X, \rho)$. We proceed to prove the inverse inequality. Since $L^1(X, \rho) < \infty$ the continuum X is regular (in the sense of Menger) and therefore for every $\epsilon > 0$ there is [8] a decomposition $X = X_1 + \cdots + X_k$ into continua such that [9]

$$\delta(X_i, \rho) < \epsilon \quad \text{for} \quad i = 1, 2, \cdots, k$$

and that $X_i \cdot X_j$ is finite if $i \neq j$. This implies that

$$L^1(X, \rho) = \sum_{i=1}^{k} L^1(X_i, \rho).$$

Since every X_i is arcwise connected it follows from the definition of ρ^* that $\delta(X_i, \rho^*) \leqq L^1(X_i, \rho)$ and hence

$$L^1(X, \rho) \geqq \sum_{i=1}^{k} \delta(X_i, \rho^*);$$

this proves that $L^1(X, \rho) \geqq L^1(X, \rho^*)$.

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[7] Saks, *loc. cit.*, p. 51. [8] I, p. 143.

[9] $\delta(A, \rho)$ stands for the diameter of the set A relative to the metric ρ.

TWIN CONVERGENCE REGIONS FOR CONTINUED FRACTIONS $b_0 + K(1/b_n)$.*

By W. J. Thron.

1. **Introduction.** This paper is concerned with the determination of twin convergence regions and the corresponding value regions for continued fractions of the form

$$(1.1) \qquad b_0 + \frac{1}{b_1} + \frac{1}{b_2} + \cdots,$$

where the elements b_n are complex numbers.

More precisely, two regions [1] B_0 and B_1 are called *twin convergence regions* if the conditions: $b_{2n} \epsilon B_0$ and $b_{2n+1} \epsilon B_1$ for all n, insure the convergence of the continued fraction (1.1) provided the following necessary conditions for convergence: $\Sigma \mid b_n \mid = \infty$, $b_{2n+1} \neq 0$ for at least one number n, are also satisfied. The principal result of this paper is Theorem 6.1. Another new convergence criterion is given in Theorem 7.3.

The methods employed in this paper are similar to those used by Leighton and Thron [1] [2] in their derivation of the parabolic convergence regions for continued fractions $1 + K(a_n/1)$. Use is again made of the theory of normal families and in particular of the generalized Stieltjes-Vitali Theorem which is stated here for later reference.[3]

Theorem 1.1. *Let $\{f_n(z)\}$ be a sequence of functions holomorphic for z in a certain region D. If $\{f_n(z)\}$ forms a normal family for all z in D, and if the sequence converges for all $z \epsilon \Delta$, where Δ is a set containing an infinite number of points having at least one limit point in the interior of D, then $\{f_n(z)\}$ converges uniformly in every closed region interior to D.*

The initial convergence theorem from which we shall derive our results is the following theorem due to Seidel [5] and Stern [6].

* Received by the editors April 12, 1943. Presented to the American Mathematical Society September 11, 1943. Acknowledgement is due to Dr. Walter Leighton for helpful suggestions in the preparation of this manuscript.

[1] The term "region" will be used both for open and closed regions.

[2] The numbers in brackets refer to the bibliography.

[3] The author is indebted to Professor S. Mandelbrojt for suggesting the use of this theorem. A proof of the theorem and a definition of the term "normal family" can be found in [2], p. 178 and p. 33, respectively.

THEOREM 1.2. *If in the continued fraction* (1.1) *the elements* b_n *are real and non-negative for all* n, *and if there exists an integer* m *such that* $b_{2m+1} \neq 0$, *then a necessary and sufficient condition for the convergence of the continued fraction* (1.1) *is the divergence of* Σb_n.

Our results can be divided into two groups. The first group consists of criteria involving twin convergence regions B_0 and B_1 both of which are bounded away from the point $z = 0$. The only result of this type known previously is a theorem by Pringsheim [4] which can be stated as follows:

If $|b_{2n}| \geqq M$, $|b_{2n+1}| \geqq N$, *where* $(1/M) + (1/N) = 1$, *then the continued fraction* (1.1) *converges.*

In Theorem 6.1 we give much more general conditions which insure that B_0 and B_1 are twin convergence regions. The corresponding value regions are determined in Theorem 4.1.

We next consider pairs of regions, which have the property that at least one of the regions has the point $z = 0$ on the boundary. The known results in this group are due to Van Vleck [7], who obtained the following result:

If the elements b_n *of the continued fraction* (1.1) *satisfy the conditions* $\alpha \leqq \arg b_{2n+1} \leqq \alpha + \pi - \epsilon$, $-\alpha - \pi + \epsilon \leqq \arg b_{2n} \leqq -\alpha$, *where* α *is an arbitrary real number and* ϵ *is positive and arbitrarily small, then a necessary and sufficient condition for the convergence of the continued fraction* (1.1) *is that the two conditions*

(i) Σb_n *diverges,*

(ii) *not all* $b_{2n+1} = 0$

hold simultaneously.

In **7** we add to this result a new two parameter family of twin convergence regions. The proof used there applies to Van Vleck's theorem as well. We further determine the value regions corresponding to all of these convergence regions.

The third case where the point $z = 0$ is in the interior of one of the twin convergence regions cannot occur. This is shown in **3**.

2. Element regions and value regions. We consider terminating continued fractions of the forms

$$b_0 + \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n}, \quad n \geqq 0, \tag{2.1}$$

and

$$b_1 + \frac{1}{b_2} + \frac{1}{b_3} + \cdots + \frac{1}{b_m}, \quad m \geqq 1, \tag{2.2}$$

whose elements are assumed to satisfy the conditions:

(i) $b_{2n} \epsilon B_0$,

(ii) $b_{2n+1} \epsilon B_1$.

The *element regions* B_0 and B_1 are arbitrary regions. At present we are not concerned with convergence questions and hence do not assume that conditions (i) and (ii) guarantee convergence of a non-terminating continued fraction whose elements satisfy these conditions.

Corresponding to the element regions B_0 and B_1 we have two *value regions* U and V. The region U contains all numbers represented by continued fractions of the form (2.1) and v contains all numbers represented by continued fractions of the form (2.2). The regions U and V are further assumed to be the smallest regions having these properties.

The following four relations are valid for the elements of these sets:

$$(2.3) \qquad u = b_0, \qquad u = b_0 + 1/v';$$

$$(2.4) \qquad v = b_1, \qquad v = b_1 + 1/u'.$$

Relations (2.3) are to be understood to mean that if b_0 and v' are arbitrary elements taken from the sets B_0 and V, respectively, then b_0 and $b_0 + 1/v'$ are elements of the set U. Relations (2.4) are to be read in an analogous manner.

We now construct, for arbitrary given sets U and V, sets B'_0 and B'_1 in such a way that relations (2.3) and (2.4) are valid for these four sets. To insure this we make the following definition.

DEFINITION 2.1. *For two given regions U and V the set B'_0 contains those points of U for which $u + 1/v'$ is an element of U, for all $v' \epsilon V$. The set B'_0 contains no other points. The set B'_1 is the totality of all those points of V for which $v + 1/u'$ is an element of V, for all $u' \epsilon U$.*

This process may lead to empty sets, or at least to one empty set. If, however, we obtain two non-null sets B'_0 and B'_1 and if U' and V' are the value regions corresponding to the sets B'_0 and B'_1 as element regions, then $U' \subset U$ and $V' \subset V$. This is easily proved by induction using relations (2.3) and (2.4). For later reference we state these results in a lemma.

LEMMA 2.1. *Let B'_0 and B'_1 be two non-null sets constructed according to* Definition 2.1 *from the regions U and V. If U' and V' are the value regions corresponding to the sets B'_0 and B'_1 as element regions, then $U' \subset U$ and $V' \subset V$.*

It is clear that the continued fractions (2.1) can have an infinite value only if either $b_1 = 0$ or if there exists a number b_1 in B_1 and a number u in U such that $b_1 = -1/u$. Thus we have the following lemma:

LEMMA 2.2. *A terminating continued fraction of the form* (2.1) *always takes on a finite value, if the regions* B_1 *and* $-1/U$ *have no point in common, and if* $z = 0 \notin B_1$.

3. Necessary conditions. The periodic continued fraction

$$b_0 + \frac{1}{b_1} + \frac{1}{b_0} + \frac{1}{b_1} + \cdots \tag{3.1}$$

is readily shown to diverge [3, p. 276] if

$$b_0 = re^{i\theta}, \qquad b_1 = \frac{4-\epsilon}{r} e^{i(\pi-\theta)},$$

where $0 < \epsilon < 4$. This consideration yields immediately the following theorem:

THEOREM 3.1. *Two element regions* B_0 *and* B_1 *can be twin convergence regions only if neither of them contains the point* $z = 0$ *in the interior. Further if the two regions are defined by*

$$b_{2n} = re^{i\theta} \in B_0, \text{ if } r \geqq f(\theta),$$
$$b_{2n+1} = re^{i\theta} \in B_1, \text{ if } r \geqq g(\theta),$$

then a necessary condition that the two regions be twin convergence regions is

$$f(\theta) \cdot g(\pi - \theta) \geqq 4. \tag{3.2}$$

We return to the continued fraction (3.1). If we choose

$$b_0 = re^{i\theta}, \qquad b_1 = g(\pi-\theta)e^{i(\pi-\theta)}$$

and let r range from $f(\theta) = 4/g(\pi - \theta)$ to infinity, then the continued fraction converges to the value

$$s = (1/2b_1)[b_0b_1 - \{b_0b_1(b_0b_1 + 4)\}^{\frac{1}{2}}].$$

Hence s takes on all values $\rho e^{i\theta}$, where $\frac{1}{2}f(\theta) \leqq \rho < \infty$. A similar result is obtained if the rôles of the elements b_0 and b_1 are interchanged. We thus arrive at the following theorem:

THEOREM 3.2. *If the element regions* B_0 *and* B_1 *are defined as in* Theorem 3.1 *and satisfy condition* (3.2), *then the value regions* U *and* V *corresponding*

to the given element regions contain, respectively, all points $u = re^{i\theta}$, $r \geqq \frac{1}{2}f(\theta)$ *and* $v = re^{i\theta}$, $r \geqq \frac{1}{2}g(\theta)$.

4. Construction of element regions bounded away from $z = 0$. We now use Definition 2.1 to construct regions B'_0 and B'_1 starting with given regions U and V.

The well known terms "complement" and "convex region" will be used with the following meaning. The set of all those points of the complex plane which do not belong to a given set A shall be called the *complement* of A. A region is *convex* if the straight line segment connecting any two points of the region is entirely contained in the region. A convex region then has the property that it lies entirely on one side of every line of support.

We choose the regions U and V in such a way that their complements are convex regions which contain the point $z = 0$ in their interior. The boundaries of the regions U and V can then be defined by two equations $r = F(\theta)$ and $r = G(\theta)$, where $F(\theta)$ and $G(\theta)$ are positive, single-valued and of period 2π.

These functions will be required to satisfy the functional equation

$$(4.1) \qquad F(\theta) \cdot G(\pi - \theta) = 1$$

and to possess second derivatives with respect to θ on the interval $[-\epsilon, 2\pi + \epsilon]$ $(\epsilon > 0)$. The requirements of convexity for both the region U and the region V are then satisfied together with condition (4.1) if the circle of curvature[4] at every point of the curve $r = F(\theta)$ contains the point $z = 0$ either in its interior or on its boundary. This condition together with the single valuedness of $F(\theta)$ insures the convexity of the region bounded by the curve $r = F(\theta)$. The curve $r = G(\theta)$ and its circles of curvature are the images of the curve $r = F(\theta)$ and its circles of curvature under the transformation of the complex plane $w = -1/z$. Thus, the circles of curvature of the curve $r = G(\theta)$ contain the point $z = 0$ either in the interior or on the boundary. This, together with the fact that the function $G(\theta)$ is single-valued, insures the convexity of the region bounded by the curve $r = G(\theta)$. The condition on the circles of curvature of the curve $r = F(\theta)$ can be written as

$$(4.2) \qquad 0 \leqq F(\theta)\,\frac{d^2F(\theta)}{d\theta^2} + F^2(\theta) < 2\left[\left(\frac{dF(\theta)}{d\theta}\right)^2 + F^2(\theta)\right].$$

We now determine the regions $1/U$ and $1/V$. The number $w = re^{i\theta}$ is an element of $1/U$ if

$$r \leqq \frac{1}{F(-\theta)} = G(\pi + \theta).$$

[4] A circle here may be a straight line, in which case we can assume that the point $z = 0$ does not lie on the line.

This is a consequence of relation (4.1). Similarly the condition for $w = re^{i\theta}$ to be an element of $1/V$ is

$$r \leqq \frac{1}{G(-\theta)} = F(\pi + \theta).$$

From Definition 2.1 it follows that those, and only those, points u of U are elements of B'_0 for which the region $u + 1/V$, that is the totality of points $u + 1/v$, $v \epsilon V$, is entirely contained in U.

For each value of θ we designate by

$$R(\theta) = 2F(\theta)e^{i\theta} + 1/V$$

the translation of the region $1/V$ effected by adding the number $2F(\theta)e^{i\theta}$ to each element of $1/V$. It follows that $R(\theta)$ has for each value of θ precisely one point, the point $F(\theta)e^{i\theta}$, in common with the boundary of U. To demonstrate this we observe that the point $F(\theta)e^{i(\pi+\theta)} = -F(\theta)e^{i\theta}$ is a point on the boundary of $1/V$. Thus $F(\theta)e^{i\theta}$ is a point on the boundary of each of the two convex regions $R(\theta)$ and the complement of U. Since the tangents to the boundaries of these two regions at this point coincide, it is clear that this point is the only point of contact of the boundaries of the two regions.

The point $2F(\theta)e^{i\theta}$ is therefore a point of B'_0 for every θ. From the preceding argument it is easily seen that $2F(\theta)e^{i\theta}$ is a point on the boundary of B'_0. For the boundary curve we therefore have the equation $r = 2F(\theta)$. An analogous argument shows that the equation of the boundary of B'_1 is $r = 2G(\theta)$.

From Lemma 2.1 it follows that the value regions corresponding to B'_0 and B'_1 as element regions are contained in U and V, respectively. By an application of Theorem 3.2 to the present regions B'_0 and B'_1 we infer that the value regions corresponding to B'_0 and B'_1 contain U and V, respectively. Hence U and V are the value regions corresponding to the element regions B'_0 and B'_1. Thus the following result has been established.

THEOREM 4.1. *Let $f(\theta)$ and $g(\theta)$ be positive functions of class C_2 and of period 2π. Further suppose that these functions satisfy the relations*

$$f(\theta) \cdot g(\pi - \theta) = 4,$$

$$0 \leqq f(\theta)\,\frac{d^2 f(\theta)}{d\theta^2} + f^2(\theta) < 2\left[\left(\frac{df(\theta)}{d\theta}\right)^2 + f^2(\theta)\right].$$

Let the element regions B_0 and B_1 be defined by the conditions

$$re^{i\theta} \,\epsilon\, B_0, \text{ if } r \geqq f(\theta);$$
$$re^{i\theta} \,\epsilon\, B_1, \text{ if } r \geqq g(\theta).$$

The value regions corresponding to these element regions are then given by

$$re^{i\theta} \epsilon U, \text{ if } r \geqq \tfrac{1}{2}f(\theta);$$
$$re^{i\theta} \epsilon V, \text{ if } r \geqq \tfrac{1}{2}g(\theta).$$

5. A preliminary convergence theorem. The apparatus developed so far will now be used to prove the following theorem which will be used later.

THEOREM 5. 1. *The continued fraction* (1. 1) *converges if* $|b_n| \geqq 2 + \epsilon$, *for all* $n \geqq 0$, *where* ϵ *is an arbitrary positive number.*

To apply Theorem 4. 1 we note that in this case we have $f(\theta) = g(\theta) = 2$. If we set $b_n = |b_n| e^{i\theta_n}$, $0 \leqq \theta_n < 2\pi$, the approximants of the continued fractions

$$(5.1) \qquad |b_0| e^{i\theta_0 z} + \frac{1}{|b_1| e^{i\theta_1 z}} + \frac{1}{|b_2| e^{i\theta_2 z}} + \cdots$$

lie in the region $|v| \geqq 1$ provided that $|b_n| \geqq 2 + \epsilon$, and z lies in the region D, where $z \epsilon D$ if $\mathfrak{J}(z) \leqq 1/2\pi \log(1 + \epsilon/2)$.

Now from Lemma 2. 2 it follows that the approximants of (5. 1) are finite for z in D. Therefore they are holomorphic functions of z. Further for z in D no function of the sequence of approximants takes on values v with $|v| < 1$. The sequence therefore constitutes a normal family of holomorphic functions for z in D. Now for $\mathfrak{R}(z) = 0$, $z \epsilon D$, the continued fraction converges by Theorem 1. 2. From Theorem 1. 1 it then follows that the continued fraction converges for all z in the interior of D. In particular, $z = 1$ is in the interior of D and hence the theorem is proved.

This theorem in somewhat sharper form was first proved by Pringsheim. It is possible that sharper results are also obtainable for some of the new regions which we proceed to determine in the next section.

6. Convergence regions bounded away from $z = 0$. As before set $b_n = |b_n| e^{i\theta_n}$. We now impose the following restrictions on the elements b_n:

$$(6.1) \qquad |b_{2n}| \geqq (1+\epsilon) f(\theta_{2n}); \qquad |b_{2n+1}| \geqq (1+\epsilon) g(\theta_{2n+1}),$$

where ϵ is an arbitrary positive constant and where the functions $f(\theta)$ and $g(\theta)$ satisfy the conditions imposed in Theorem 4. 1.

Consider the continued fraction

$$(6.2) \qquad b_0 z + \frac{1}{b_1 z} + \frac{1}{b_2 z} + \cdots,$$

where the numbers b_n are supposed to satisfy conditions (6. 1). For z in the region D, where $z \epsilon D$ if $|z| \geqq 1/(1+\epsilon)$, the approximants of (6. 2) are finite,

by Lemma 2. 2, and do not take on any value $v = re^{i\theta}$, $r < f(\theta)$. The sequence of approximants therefore constitutes a normal family of holomorphic functions for all values of the variable z in the region D. Now let M be the minimum value assumed by $f(\theta)$ and $g(\theta)$. Clearly $M > 0$. For $|z| > (2+\epsilon)/M$ the continued fraction then converges by Theorem 5. 1. An application of Theorem 1. 1 then insures the convergence of the continued fraction for $|z| > 1/(1+\epsilon)$ and in particular for $z = 1$. The following theorem has now been established:

THEOREM 6. 1. *Let the regions B_0 and B_1 be defined by:*

$$r \cdot e^{i\theta} \epsilon B_0 \text{ if } r \geqq (1+\epsilon) f(\theta), \qquad r \cdot e^{i\theta} \epsilon B_1 \text{ if } r \geqq (1+\epsilon) g(\theta),$$

where ϵ is an arbitrarily small positive number and the functions $f(\theta)$ and $g(\theta)$ are positive and finite in the interval $[0, 2\pi]$. A necessary condition for B_0 and B_1 to be twin convergence regions is

$$f(\theta) g(\pi - \theta) \geqq 4.$$

Let $g(\theta) = 4/f(\pi - \theta)$, then the regions B_0 and B_1 are twin convergence regions if the complements of both regions are convex.

As corollaries of the principal theorem we have the following results concerning circular convergence regions.

COROLLARY 6. 1. *Let the real numbers a and c satisfy the inequality $a > c \geqq 0$, let γ be an arbitrary angle and ϵ an arbitrarily small positive constant. Then the continued fraction* (1. 1) *converges if for all $n \geqq 0$*

$$| b_{2n} - ce^{i\gamma} | \geqq a + \epsilon,$$

and

$$\left| b_{2n+1} - \frac{4c \cdot e^{-i\gamma}}{a^2 - c^2} \right| \geqq \frac{4a}{a^2 - c^2} + \epsilon.$$

COROLLARY 6. 2. *Let c be an arbitrary real number and ϵ an arbitrarily small positive constant. Then the continued fraction* (1. 1) *converges if for all $n \geqq 0$*

$$| b_n - c | \geqq \sqrt{c^2 + 4} + \epsilon.$$

There are many other regions that satisfy the conditions of Theorem 6. 1; as a final example we mention that the curve $r = f(\theta)$ might be the envelope of the family of circles with centers on an ellipse around the origin passing through the point $z = 0$.

7. Convergence regions having the point $z = 0$ on the boundary. It follows from Theorem 3.1 that there cannot exist twin convergence regions one or both of which contain the point $z = 0$ in the interior. The only case that remains to be considered is the case in which the point $z = 0$ lies on the boundary of at least one of the element regions. Proceeding as in **4** we consider a pair of value regions, one of which now has the point zero on the boundary. We again assume, however, that the complements of the two value regions are convex and that the product of no two numbers u and v, $u \epsilon U$ and $v \epsilon V$, satisfies the condition $uv = \delta i$, $0 < \delta < 1$.

It can be shown that the new boundary conditions are so restrictive that they, together with the old conditions, are satisfied only by the regions bounded by the curves

$$r = a \cos(\theta - \gamma), \qquad r = (1/a) \sec(\pi - \theta - \gamma);$$

or by the half-planes bounded by the lines

$$\theta = \alpha, \theta = \pi + \alpha; \quad \theta = -\alpha, \theta = -\alpha - \pi.$$

To show this it is sufficient to note that at least one of the boundaries of the two regions is a line which approaches the same straight line asymptotically as it goes to ∞ in both directions. Such a line can then bound a convex region only if it is the straight line itself.

An argument quite analogous to that used in proving Theorem 4.1 then leads to the following theorems:

THEOREM 7.1. *Let the element regions B_0 and B_1 be defined as follows:*

$re^{i\theta} \epsilon B_0$, *if* $r \geqq a\cos(\theta - \gamma)$ *for* $\gamma - \pi/2 \leqq \theta \leqq \gamma + \pi/2$, $r \geqq 0$ *otherwise*;
$re^{i\theta} \epsilon B_1$, *if* $r \geqq (4/a)\sec(\pi - \theta - \gamma)$ *for* $-\gamma - \pi/2 \leqq \theta \leqq -\gamma + \pi/2$.

The value regions corresponding to these element regions are:

$re^{i\theta} \epsilon U$, *if* $r \geqq \frac{1}{2}a\cos(\theta - \gamma)$ *for* $\gamma - \pi/2 \leqq \theta \leqq \gamma + \pi/2$, $r \geqq 0$ *otherwise*;
$re^{i\theta} \epsilon V$, *if* $r \geqq (2/a)\sec(\pi - \theta - \gamma)$ *for* $-\gamma - \pi/2 \leqq \theta \leqq -\gamma + \pi/2$.

THEOREM 7.2. *Let B_0 be the half-plane $re^{i\theta} \epsilon B_0$, if $\alpha \leqq \theta \leqq \alpha + \pi$, and B_1 the half-plane $re^{i\theta} \epsilon B_1$, if $-\alpha - \pi \leqq \theta \leqq -\alpha$. Then $U = B_0$ and $V = B_1$.*

To prove that these regions are twin convergence regions we use an argument similar to that of **6**. We shall carry the proof through with the regions of Theorem 7.1, but shall take as B_0 the region there called B_1, and accordingly B_1 will now be the region there denoted by B_0. This introduces a difficulty which does not appear in the earlier case. The difficulty is that Lemma 2.2 does not apply here. To obtain its equivalent we shall now demand that not all the b_{2n+1} in the continued fraction (1.1) vanish. Let b_{2N+1}

be the first element with odd subscript which does not vanish. For all $n \geqq 2N+1$ the n-th approximants of (1.1) are then finite, if the elements b_n lie in the regions B_0 and B_1, respectively. This is seen as follows. In order that

$$b_0 + \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n}, \quad n \geqq 2N+1$$

be infinite it is necessary, with the given conditions on the elements b_n, that $b_1 = 0$ and

$$b_2 + \frac{1}{b_3} + \cdots + \frac{1}{b_n} = \infty.$$

This implies $b_3 = 0$ and

$$b_4 + \frac{1}{b_5} + \cdots + \frac{1}{b_n} = \infty.$$

Proceeding in this manner we can show that an n-th approximant is infinite only if $b_{2m+1} = 0$ for all m for which $2m+1 \leqq n$.

We set

$$b_{2n} = | b_{2n} | e^{i\gamma} e^{i\theta_{2n}}, \ | \theta_{2n} | \leqq \pi; \quad b_{2n+1} = | b_{2n+1} | e^{-i\gamma} e^{i\theta_{2n+1}}, \ | \theta_{2n+1} | \leqq \pi$$

and consider the continued fraction

$$(7.1) \qquad | b_0 | e^{i\gamma} e^{i\theta_0 z} + \frac{1}{| b_1 | e^{-i\gamma} e^{i\theta_1 z}} + \frac{1}{| b_2 | e^{i\gamma} e^{i\theta_2 z}} + \cdots,$$

where the numbers b_n are required to satisfy the following conditions:

$$| \theta_{2n} | \leqq \pi/2 - \delta, \ | b_{2n} | \geqq (1+\epsilon)(4/a)\sec(| \theta_{2n} | + \delta)$$

and

$$| b_{2n+1} | \geqq (1+\epsilon) a \cos(\pi - | \theta_{2n+1} | - \delta), \text{ for } | \theta_{2n+1} | > \pi/2 - \delta;$$
$$| b_{2n+1} | \geqq 0 \text{ otherwise.}$$

The constants ϵ and δ are here arbitrarily small positive numbers. Under these conditions there exists a region D containing the line segment $[0, 1]$ in its interior, such that for all values of the variable z in D the elements of the continued fraction (7.1) are in the regions B_0 and B_1, respectively.

This, together with the assumption that not all b_{2n+1} vanish, insures that the approximants of (7.1), from a certain n on, form a normal family of holomorphic functions for all z in D. For $\mathfrak{R}(z) = 0$, $z \in D$, the continued fraction (7.1) converges if $\Sigma | b_n | = \infty$. This is clear since the continued fraction (7.1) is then equivalent [3, p. 196] to

$$| b_0 | e^{-\theta_0 y} + \frac{1}{| b_1 | e^{-\theta_1 y}} + \frac{1}{| b_2 | e^{-\theta_2 y}} + \cdots,$$

and this continued fraction converges under the given conditions by Theorem 1.2 as the quantities $e^{-\theta_n y}$ are bounded away from zero uniformly with respect to n.

The convergence of (7.1) then follows from Theorem 1.1 for all z in D and in particular for $z = 1$. With some changes in notation our result can then be stated as follows:

THEOREM 7.3. *Let δ be an arbitrarily small positive constant, c an arbitrary positive number and γ an arbitrary angle. Further let all elements b_{2n+1} $(n \geqq 0)$ lie outside the family of circles*

$$| z - ce^{i(\pi-\gamma+\eta)} | < c,$$

where η varies from $-\delta$ to δ; and let all b_{2n} $(n \geqq 0)$ lie in the angular opening which is the intersection of the two half-planes

$$\Re(ze^{-i(\gamma+\delta)}) \geqq 2/c \quad \textit{and} \quad \Re(ze^{-i(\gamma-\delta)}) \geqq 2/c.$$

The continued fraction

$$b_0 + \frac{1}{b_1} + \frac{1}{b_2} + \cdots$$

then converges if and only if not all b_{2n+1} vanish. The rôles of the b_{2n} and b_{2n+1} may be interchanged.

Van Vleck's Theorem, as stated in **1**, can be proved in an analogous manner.

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CONCERNING TRIODIC CONTINUA.*[1]

By R. H. Sorgenfrey.

This paper is primarily a study of certain internal properties of continua, especially those which pertain to the decomposability of continua into three subcontinua in several specified ways. To this end there are presented in the first section definitions of eight types of continua which will be referred to as triods. Several of these definitions have been given before by other writers and are to be found in the literature. In the second section the structure of these triods is investigated with regard to what will be called their nuclei. In the last section some theorems concerning continua which are characterized, in part, by failing to be triods are presented. This group includes several conditions under which a continuum will be irreducible between two points.

The reader will notice that the definitions of the paper could be restated to pertain to decompositions of continua into any finite number of parts, rather than three, and that most of the theorems, with corresponding modifications, would remain valid.

Throughout the paper it will be assumed that S, the set of all points, is a Moore space, that is, one which satisfies Axiom 0 and the first three parts of Axiom 1 of R. L. Moore's *Foundations of Point Set Theory*.[2] All of the theorems are thus valid in any metric space.

1. A consideration of several types of triods. Of the definitions given below, those of triods of types 3, 7, and 8 are due to R. L. Moore and the one of type 4 is due to G. T. Whyburn. Definitions of triods of types 5 and 6 have been given to obtain classes of continua which approximate simple triods more nearly than do triods of types 3 and 4, but which still do not necessarily fulfill conditions as strict as those imposed by the definitions of types 7 and 8. The main justification for the introduction of the definitions of triods of types 1 and 2 is to be found in Theorems 1.7 and 1.8 of this section. The conditions

* Received April 6, 1943.

[1] Presented in part to the American Mathematical Society, February 28, 1939 and December 30, 1940.

[2] *American Mathematical Society Colloquium Publications*, vol. 13 (New York, 1932). Hereinafter this book will be referred to as *Foundations*. The reader is referred to this book for the meanings of terms used but not specifically defined in this paper.

imposed by these definitions are more lax even than those of the definition of the type 3 triod.

Definitions. The continuum T is said to be a triod of type n provided it satisfies the n-th of the following conditions.

Condition 1. T is the sum of three continua which have a point in common and such that no one of them is a subset of the sum of the other two.

Condition 2. T is the sum of three continua such that the common part of all of them is both a proper non-vacuous subset of each of them and the common part of each two of them.

Condition 3. T is the sum of three continua such that the common part of all of them is both a proper non-vacuous subcontinuum of each of them and the common part of each two of them.[3]

Condition 4. T contains a continuum N and three points X, Y, and Z, not belonging to N, and is the sum of three continua irreducible about $N + X$, $N + Y$, and $N + Z$, respectively, and such that N is the common part of each two of them.[4]

Condition 5. T contains four points X, Y, Z, and O and a continuum N containing none of the points X, Y, and Z and is the sum of three continua irreducible from O to X, Y, and Z, respectively, and such that the common part of each two of them is N.

Condition 6. T contains a continuum N and three points X, Y, and Z and is the sum of three continua irreducible from each point of N to X, Y, and Z, respectively, and such that the common part of each two of them is N.

Condition 7. *T contains four points* X, Y, Z, and O and is the sum of three continua irreducible from O to X, Y, and Z, respectively, and such that O is the common part of each two of them.[5]

[3] R. L. Moore, "Concerning triodic continua in the plane," *Fundamenta Mathematicae*, vol. 13 (1929), p. 262. In this paper Moore does not specifically define the term triod, but defines a continuum to be triodic or atriodic according as it contains or does not contain a type 3 triod.

[4] G. T. Whyburn, "Concerning collections of cuttings of connected point sets," *Bulletin of the American Mathematical Society*, vol. 35 (1929), p. 104. In this paper Whyburn refers to a type 4 triod as the "analog of a triod" of Moore, that is, of a type 7 triod.

[5] R. L. Moore, "Concerning triods in the plane and the junction points of plane continua," *Proceedings of the National Academy of Sciences*, vol. 14 (1928), p. 85.

Condition 8. T contains a point O and is the sum of three arcs each having O as an end point and such that the common part of each two of them is O.[6]

It follows immediately from the above definitions that a triod of type n $(n = 2, 3, \cdots, 8)$ is a triod of type $n - 1$; and simple examples exist to show that a triod of type n $(n = 1, 2, \cdots, 7)$ is not necessarily a triod of type $n + 1$. In case T is a compact continuous curve,[7] the definitions of triods of types 6, 7, and 8 are all equivalent; no other two definitions are equivalent in this case, however, although a triod of type 5 is necessarily a triod of type 8 or the sum of four arcs emanating from a point Q, each two of them having only Q in common.

THEOREM 1. 1. *In order that a continuum T be a type 2 triod it is necessary and sufficient that there exist three mutually exclusive open subsets of T such that the sum of no two of them disconnects T.*

Proof. To prove that the condition is necessary, suppose that T is a type 2 triod. Then, by definition, T is the sum of three continua H, K, and L such that $H \cdot K = K \cdot L = H \cdot L$ and $H \cdot K$ is a proper subset of H, K, and L. Then $H - H \cdot K$, $K - H \cdot K$, and $L - H \cdot K$ are three mutually exclusive open subsets of T. Moreover, T is not disconnected by the omission of any two of them since if the sum of any two of them is subtracted from T, the remainder is one of the continua H, K, and L.

To prove that the condition is sufficient, suppose that D, E, and F are mutually exclusive open subsets of the continuum T such that the omission of any two of them leaves T connected. Then $T - (D + E)$, $T - (E + F)$, and $T - (D + F)$ are three continua whose sum is T, and the common part of each two of them is $T - (D + E + F)$. This is non-vacuous, since T is not the sum of three mutually exclusive open subsets, and is clearly a proper subset of each of the three continua.

THEOREM 1. 2. *In order that a continuum T be a type 3 triod it is necessary and sufficient that there exist three mutually exclusive open subsets of T such that their sum does not disconnect T.*[8]

A proof very similar to that of the preceding theorem may be given for this one if use is made of the fact that if the sum of three mutually exclusive

[6] R. L. Moore, *Foundations*, p. 250.
[7] That is, a compact, connected im kleinen continuum.
[8] This theorem was suggested by W. B. Coleman.

open subsets of a continuum fail to disconnect it, then the sum of no two of them will disconnect it.

THEOREM 1.3. *Every unicoherent*[9] *type* 1 *triod is a type* 3 *triod.*

Proof. Suppose that T is a unicoherent triod of type 1. Then it is the sum of three continua H, K, and L with a common point P and such that no one of them is a subset of the sum of the other two. Since T is unicoherent, the point sets $H \cdot (K + L)$, $K \cdot (H + L)$, and $L \cdot (H + K)$ are continua. Hence, since they all contain P, their sum is a continuum N. The continuum N is a proper subset of each of the continua $N + H$, $N + K$, and $N + L$. Furthermore N is their common part and their sum is T. Hence T is a type 3 triod.

The property of triods stated in the following theorem is useful in the study of irreducible continua.

THEOREM 1.4. *No triod is irreducible between any two points.*

THEOREM 1.5. *Every compact type* 5 *triod is irreducible about the four points* X, Y, Z, *and* O (*using the notation of the definition*).

Proof. Suppose that there exists a proper subcontinuum M of T which contains the points X, Y, Z, and O. Denote by C the component of $M - M \cdot N$ containing X. Let H, K, and L be the irreducible continua of the definition which contain X, Y, and Z, respectively. Since $H - N$, $K - N$ and $L - N$ are mutually separated sets, C is a subset of $H - N$. By Theorem 40 of Chapter 1 of *Foundations*, $\bar{C}$ intersects N. Hence, unless C contains $H - N$, $\bar{C} + N$ is a proper subcontinuum of H containing both X and O, which is impossible since H is irreducible from X to O. Therefore M contains $H - N$. By similar arguments it may be shown to contain $K - N$ and $L - N$ and hence $T - N$. The set $H - N$ does not have O as a limit point for if it did $\overline{H - N}$ would be a subcontinuum of H containing X and O and hence would be H. But then M, which contains $H - N$, would contain H and, therefore, N, which is impossible since M is, by supposition, a proper subcontinuum of T. Similarly neither of the sets $K - N$ nor $L - N$ has O as a limit point and hence $T - N$ does not. Denote by F the component of $M - \overline{(T - N)}$ which contains O. The continuum $\bar{F}$ intersects $\overline{T - N}$ and hence one of the continua $\overline{H - N}$, $\overline{K - N}$, and $\overline{L - N}$. If $\bar{F}$ intersects $\overline{H - N}$, $\bar{F} + (\overline{H - N})$

[9] A continuum M is said to be unicoherent provided the common part of every two continua whose sum is M is connected.

is a subcontinuum of H which contains X and O and which, therefore, must be H. But then $\bar{F}$ contains $N - N \cdot (\overline{T - N})$ and M is not a proper subcontinuum of T. It follows in a similar manner that $\bar{F}$ cannot intersect $\overline{K - N}$ or $\overline{L - N}$. This contradiction establishes the theorem.

THEOREM 1. 6. *Every compact type* 6 *triod is irreducible about the three points X, Y, and Z (using the notation of the definition).*

Proof. Suppose that M is a subcontinuum of T which contains X, Y and Z. As in the preceding theorem it may be shown that if H, K, and L are the three continua of the definition, then $H - N$, $K - N$, and $L - N$ are connected subsets of M. Since $\overline{H - N}$ is a subcontinuum of H containing X and a point of N, it must be H and hence M contains N. Hence M is T.

THEOREM 1. 7. *If a locally compact continuous curve contains a type* 1 *triod it contains a type* 8 *triod.*

This theorem follows immediately from a theorem of F. B. Jones [10] to the effect that a locally compact continuous curve which contains no simple triod is a simple continuous curve. Jones has also shown that in a complete Moore space [11] the same conclusion follows even if the hypothesis of local compactness is omitted. Hence, in case S is a complete Moore space, the words "locally compact" may be omitted from the hypothesis of Theorem 1. 7.

The following theorem suggests a simplification of the definition of the term "triodic" [3] for compact continua.

THEOREM 1. 8. *Every compact type* 1 *triod contains one of type* 3.

Proof. Suppose that T is a compact type 1 triod. Consider T to be space.[12] By hypothesis T is the sum of three continua H, K, and L having a point P in common and such that no one of them is a subset of the sum of the other two. Denote the closed set $H \cdot K + K \cdot L + H \cdot L$ by N. Three cases arise. Case 1. Suppose that N is connected. Then T is the sum of the three continua $H + N$, $K + N$, and $L + N$. But their common part and

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[10] F. B. Jones, "Concerning the boundary of a complementary domain of a continuous curve," *Bulletin of the American Mathematical Society*, vol. 45 (1939), pp. 428-429.

[11] A complete Moore space is one which satisfies Axioms 0 and 1 of *Foundations*.

[12] Suppose that K is a subset of a Moore space S. If K is regarded as space and the term "region" is taken to mean the common part of K and a region in the space S, then with this interpretation of "point" and "region," K is a Moore space and limit point is invariant under the change.

the common part of each two of them is the continuum N and N is a proper subset of each of them. Hence in this case T not only contains but is a type 3 triod. Case 2. Suppose that N is the sum of two mutually exclusive continua U and V such that U contains P. Denote by X, Y, and Z points belonging to H, K, and L, respectively, but not to U. There exists a domain D containing U and such that $\bar{D}$ does not intersect $V + X + Y + Z$. Denote by H', K', and L' the components of $H \cdot \bar{D}$, $K \cdot \bar{D}$, and $L \cdot \bar{D}$, respectively, which contain P. Since H', K', and L' intersect $\bar{D} - D$, the sets $H' - H' \cdot U$, $K' - K' \cdot U$, and $L' - L' \cdot U$ exist. Since these three sets are mutually exclusive, it follows that the sum of the continua $H' + U$, $K' + U$, and $L' + U$ is a type 3 triod. Case 3. Suppose that N is the sum of three mutually exclusive closed point sets U, V, and W such that U contains P. Since all the continua H, K, and L intersect U and two of them intersect V and two of them intersect W, it follows that one of them intersects all three of the sets U, V, and W. Suppose that H does this. Let Q and O be points of $H \cdot V$ and $H \cdot W$, respectively. There exist domains D_U, D_V, and D_W containing U, V, and W, respectively, and such that $\bar{D}_U$, $\bar{D}_V$, and $\bar{D}_W$ are mutually exclusive. Denote by H_P the component of $H \cdot \bar{D}_U$ containing P, by H_Q the component of $H \cdot \bar{D}_V$ containing Q, and by H_O the component of $H \cdot \bar{D}_W$ containing O. Since it contains a point of $\bar{D}_U - D_U$, H_P is not a subset of N. Therefore it is not a subset of $K + L$. Similarly neither H_Q nor H_O is a subset of $K + L$. Since H_P, H_Q, and H_O are mutually exclusive, it follows that the sum of the continua $H_P + K + L$, $H_Q + K + L$, and $H_O + K + L$ is a type 3 triod.

COROLLARY. *Every compact triod of type* 1 *contains one of type* 4.

Simple examples exist to show that a continuum which contains a type n triod ($n = 4, 5, 6, 7$) does not necessarily contain a triod of type $n + 1$. That Theorem 1.8 does not remain true if the condition of compactness is omitted from its hypothesis is shown by the following examples. C. Kuratowski has described [13] a plane indecomposable continuum W which contains no type 1 triod and which has two composants which are quite similar in a sense to rays. Denote the end points of these composants by P and Q, and let PX and QY be arcs which are subsets of W. Two examples will be constructed. Example 1. Let F_1 and F_2 be two topological transformations of W into W_1 and W_2, respectively, such that if the images of P under F_1 and F_2 are P_1 and P_2, respectively, etc., then $P_1 = P_2$ and $Q_1 = Q_2$ but W_1 and W_2 have no

[13] C. Kuratowski. "Théorie des continus irréductibles entre deux points," I, *Fundamenta Mathematicae*, vol. 3 (1922), pp. 215-217.

other points in common. Let X_1X_2 denote the arc which is the sum of the arcs P_1X_1 and P_2X_2. Let R be a point set which is topologically equivalent to a ray, which does not intersect $W_1 + W_2$, which has each point of X_1X_2 as a limit point, but which has no other limit point not belonging to itself. Let M denote the point set $W_1 + W_2 + R - (X_1 + X_2)$. If M is considered as space, it is a type 1 triod which does not contain a type 2 triod. Example 2. Let F_1, F_2, and F_3 be topological transformations of W into W_1, W_2, and W_3, respectively, such that if the images of P under F_1, F_2, and F_3 are P_1, P_2, and P_3, respectively, etc., then the arcs P_1X_1, P_2X_2, and P_3X_3 coincide, and the arcs Q_1Y_1, Q_2Y_2, and Q_3Y_3 coincide, but no two of the continua W_1, W_2, and W_3 have any other point in common. Let M denote the point set $W_1 + W_2 + W_3 - (X_1 + Y_1)$. If M is considered as space, it is a type 2 triod which contains no type 3 triod. These examples show that if the word "compact" is omitted from the hypothesis of Theorem 1.8, the resulting proposition is not true even if the hypothesis is strengthened by replacing the words "type 1" by the words "type 2" or if the conclusion is weakened by replacing the words "type 3" by the words "type 2."

THEOREM 1.9. *If a compact continuous curve contains a type* 1 *triod then it is itself a type* 1 *triod.*

Proof. Let M be a compact continuous curve which contains a type 1 triod. By Theorem 1.7 it contains three arcs OA, OB, and OC each two of which have only the point O in common. Since M is locally connected, there exist three connected open subsets U, V, and W of M containing A, B, and C, respectively, such that $\bar{U}$, $\bar{V}$, and $\overline{W}$ are mutually exclusive, and such that $\bar{U}$ contains no point of $OB + OC$, $\bar{V}$ contains no point of $OA + OC$, and $\overline{W}$ contains no point of $OA + OB$. Denote by A', B', and C' the first points of OA, OB, and OC, respectively, which belong to $\bar{U}$, $\bar{V}$, and $\overline{W}$; and let N be the sum of O and the segments OA', OB', and OC' of these arcs. If Z is U, V, or W, let G_Z denote the collection of all components X of $M - (\bar{U} + \bar{V} + \overline{W})$ such that X has a limit point in $\bar{Z}$. Then $\bar{U} + G^*_U$,[14] $\bar{V} + G^*_V$, and $\overline{W} + G^*_W$ are continua whose sum is M. Since N is a connected subset of $M - (\bar{U} + \bar{V} + \overline{W})$ which has the points A', B', and C' of $\bar{U}$, $\bar{V}$, and $\overline{W}$, respectively, as limit points, it follows that the component of $M - (\bar{U} + \bar{V} + W)$ which contains O belongs to each of the collections G_U, G_V, and G_W and therefore that the continua $\bar{U} + G^*_U$, $\bar{V} + G^*_V$, and $\overline{W} + G^*_W$ have the point O in common. Since, moreover, no one of these

[14] If G is a collection of point sets, G^* is the point set which is the sum of the elements of G.

continua is a subset of the sum of the other two, it follows that M is a type 1 triod.

COROLLARY. *Every compact continuous curve is either an arc, a simple closed curve, or a type* 1 *triod.*

The reader may verify the fact that the number of topologically distinct compact continuous curves which fail to be type 3 triods is very small. Every compact acyclic continuous curve, for example, which is not an arc is a type 3 triod.

2. Concerning the structure of triods. In this section and that which follows the terms "triod" and "simple triod" will be used in place of "type 3 triod" and "type 8 triod," respectively. Triods of other types will be referred to as in **1**. A continuum will be said to be "triodic" or "atriodic" according as it contains or does not *contain* a type 3 triod. It will be convenient to speak of a triodic decomposition of a triod T into three continua H, K, and L. This will mean that T is the sum of the continua H, K, and L and that their common part is both a proper non-vacuous subcontinuum of each of them and the common part of each two of them.

Consider a simple triod which is the sum of the arcs OA, OB, and OC. Moore refers to the point O as the emanation point of the triod.[5] It is the purpose of this section to define and discuss subsets (not necessarily points) of more general types of triods which have some analogous properties.

Definition. The subcontinuum N of a triod T is called a nucleus of T if there exists a triodic decomposition of T into three continua H, K, and L such that $H \cdot K = N$ and such that there does not exist a triodic decomposition of T into three continua H', K', and L' which are subsets of H, K, and L, respectively, and such that $H' \cdot K'$ is a proper subcontinuum of N.

THEOREM 2.1. *Every compact triod has at least one nucleus.*

Proof. Suppose that there exists a compact triod T which has no nucleus. Consider T as space. Since T is compact, it is completely separable, and there exists a countable collection $D_1, D_2, D_3, \cdots$ of domains such that if P is a point of a domain D, there exists an integer n such that D_n contains P and is a subset of D. There exists a triodic decomposition of T into three continua H_0, K_0, and L_0. Denote the continuum $H_0 \cdot K_0$ by N_0. By supposition, N_0 is not a nucleus and therefore there exists a triodic decomposition of T into continua H, K, and L which are subsets of H_0, K_0, and L_0, respectively, and such that $H \cdot K = N$ is a proper subcontinuum of N_0. Let P be a point of

$N_0 - N$. There exists an integer k such that D_k contains P and is a subset of $T - N$. Let n_1 denote the smallest integer n such that D_n contains a point of N_0 and such that there exists a triodic decomposition of T into continua H, K, and L which are subsets of H_0, K_0, and L_0, respectively, and such that $H \cdot K$ is a subset of $N_0 - N_0 \cdot D_n$. Let H_1, K_1, and L_1 constitute such a triodic decomposition of T and denote $H_1 \cdot K_1$ by N_1. By supposition N_1 is not a nucleus, and the above process may be repeated. Successive repetitions of the process will give rise to sequences $H_0, H_1, H_2, \cdots$; $K_0, K_1, K_2, \cdots$; $L_0, L_1, L_2, \cdots$; and $N_0, N_1, N_2, \cdots$ of continua such that for each positive integer n, T is triodicly decomposable into the continua H_n, K_n, and L_n; $H_n \cdot K_n = N_n$; and further H_n, K_n, L_n, and N_n are subsets of H_{n-1}, K_{n-1}, L_{n-1}, and N_{n-1}, respectively. The common part of the elements of each of these four sequences is a continuum. Let these common parts be, respectively, H, K, L, and N. It follows that the continua H, K, and L form a triodic decomposition of T and that $H \cdot K = N$. By the initial supposition N is not a nucleus. Hence there exists a triodic decomposition of T into continua H', K', and L' which are subsets of H, K, and L, respectively, and such that $H' \cdot K' = N'$ is a proper subcontinuum of N. Let P be a point of $N - N'$. There exists an integer k such that D_k contains P and is a subset of $T - N'$. But this leads to a contradiction because (1) k is not less than n_k, for n_k is the smallest integer n such that D_n contains a point of N_{k-1} and such that T is triodicly decomposable into three continua which are subsets of H_{k-1}, K_{k-1}, and L_{k-1} and such that the common part of these three continua is a subset of $N_{k-1} - N_{k-1} \cdot D_n$; (2) k is not equal to n_n, for D_{n_k} contains no point of N_k and therefore not P; and (3) k is not greater than n_k since the integers $n_1, n_2, n_3, \cdots$ are distinct.

By a modification of the proof of Theorem 2.1 the following theorem may be established.

THEOREM 2.2. *If the triod T is triodicly decomposable into the continua H, K, and L such that $H \cdot K$ is compact, then it is triodicly decomposable into continua H', K', and L' which are subsets of H, K, and L, respectively, and such that $H' \cdot K'$ is a nucleus of T.*

That the two preceding theorems do not follow if the condition of compactness is omitted is shown by the following example. Let I be a straight line interval in the plane and let O be its mid-point. Let U, V, and W be point sets each of which is topologically equivalent to a ray, such that each point of I is a limit point of each of them, such that each of the sets $I + U$, $I + V$, and $I + W$ is closed, and such that I, U, V, and W are mutually

exclusive. Let $T = I + U + V + W - O$. Then, considered as space, T is a triod which has no nucleus.

THEOREM 2. 3. *If T is a compact triod and the subcontinuum I of T is irreducible between two mutually exclusive nuclei N_1 and N_2 of T then there exists a nucleus of T which contains I and is a subset of $I + N_1 + N_2$.*[15]

Proof. Since N_1 and N_2 are nuclei of T, there exists for each i $(i = 1, 2)$ a triodic decomposition of T into continua H_i, K_i, and L_i such that $H_i \cdot K_i = N_i$. Since N_1 and N_2 are mutually exclusive, N_2 lies in one of the sets $H_1 - N_1$, $K_1 - N_1$, and $L_1 - N_1$. Suppose that it lies in $H_1 - N_1$ and suppose that N_1 lies in $H_2 - N_2$. The continuum I is then a subset of $H_1 \cdot H_2$. Since, moreover, neither K_2 nor L_2 intersects N_1, it follows that $K_2 + L_2$ is a subset of $H_1 - N_1$. Similarly $K_1 + L_1$ is a subset of $H_2 - N_2$. The sum of the three continua $H' = L_1 + I + N_2$, $K' = L_2 + I + N_1$, and $L' = K_1 + K_2 + H_1 \cdot H_2$ is T, and the common part of each two of them is the continuum $I + N_1 + N_2$; they form, therefore, a triodic decomposition of T. It follows from Theorem 2. 2 that there exists a triodic decomposition of T into continua H, K, and L which are subsets of H', K', and L', respectively, and such that $H \cdot K = N$ is a nucleus of T. Suppose that N contains no point of N_1. Then, since N is a subset of $I + N_1 + N_2$ and, therefore, does not intersect $K_1 - N_1$ or $L_1 - N_1$, the connected set $K_1 + L_1$ is a subset of just one of the mutually separated sets $H - N$, $K - N$, and $L - N$. But this is impossible since $H - N$ contains $L_1 - N_1$ and $L - N$ contains $K_1 - N_1$. Hence N contains a point of N_1. Similarly N contains a point of N_2; and since I is irreducible between N_1 and N_2, it follows that N contains I. This establishes the theorem.

That it does not follow from the hypothesis of Theorem 2. 3 that I is a nucleus of T nor yet that $I + N_1 + N_2$ is a nucleus is shown by the following example. In a Cartesian plane let U be the graph of the equation $y = \sin(1/x)$ for $0 < x \leq 1/\pi$, and let V be the interval of the y-axis whose end points have ordinates 2 and -2. Let E be an arc whose end points have co-ordinates $(0, 1)$ and $(0, -1)$ and which lies, except for its end points, wholly to the left of the y-axis, and let F be the interval of the x-axis whose end points have abscissas $1/\pi$ and $2 + 1/\pi$. Let W be the graph of the equation $x = 1 + 1/\pi + \sin(1/y)$ for $1 \geqq y > 0$ and $0 > y \geqq -1$. Then if T is the sum of the point sets U, V, W, E, and F, T is a triod which has the continua E and F as two of its nuclei. The continuum $\bar{U}$ is irreducible between E and F, but neither $\bar{U}$ nor $\bar{U} + E + F$ is a nucleus of T.

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[15] This theorem was suggested by R. L. Swain.

Nuclei of triods as general as those of type 3 are very little restricted as to number or form. A triod which consists of a circle plus its interior, for example, has uncountably many nuclei and these may be continua as different in form as an indecomposable continuum and a simple triod. For more special types of triods, however, the nuclei behave more as would be intuitively expected.

THEOREM 2.4. *No compact type 5 triod has two mutually exclusive nuclei.*

Proof. Let T be a compact type 5 triod. By definition there exist a continuum N, a point O of N, and three points X, Y, and Z not belonging to N such that T is the sum of three continua H, K and L irreducible from O to X, Y, and Z, respectively, and such that the common part of each two of them is N. Suppose that there exist two triodic decompositions of T into continua H_1, K_1, and L_1 and H_2, K_2, and L_2, respectively, such that $H_1 \cdot K_1 = N_1$ and $H_2 \cdot K_2 = N_2$ are mutually exclusive. The continuum N_1 lies wholly in $H_2 - N_2$, $K_2 - N_2$, or $L_2 - N_2$ and N_2 lies wholly in $H_1 - N_1$, $K_1 - N_1$, or $L_1 - N_1$. Suppose that $H_2 - N_2$ contains N_1 and that $H_1 - N_1$ contains N_2. The set $T - H_1 \cdot H_2$ is the sum of the mutually separated sets $K_1 - N_1$, $L_1 - N_1$, $K_2 - N_2$, and $L_2 - N_2$. It follows from Theorem 1.5 that each of these sets contains one of the points X, Y, Z, and O. Suppose that X, Y, Z, and O belong to $K_1 - N_1$, $L_1 - N_1$, $K_2 - N_2$, and $L_2 - N_2$, respectively. The set N contains a point of N_1 because otherwise $K_1 + L_1$ would be a connected subset of $T - N$ containing both X and Y. Since N contains O and N_2 separates N_1 from O in T, it follows that N contains a point of N_2. It therefore, by Theorem 1.5, contains $H_1 \cdot H_2 - (N_1 + N_2) = T - (K_1 + L_1 + K_2 + L_2)$. Since (1) $K_1 + L_1$ and $K_2 + L_2$ are mutually separated, (2) X and Y belong to $K_1 + L_1$, and (3) $H - N$ and $K - N$ are connected subsets of $(K_1 + L_1) + (K_2 + L_2)$, it follows that $H + K - N$ does not intersect $K_2 + L_2$ and therefore that $K_2 + L_2$ is a subcontinuum of L. Hence, since $K_2 + L_2$ contains both Z and O, it is L. But this is impossible because it does not contain N_1 and hence not all of N.

THEOREM 2.5. *Every compact type 6 triod has just one nucleus.*

Proof. Suppose that T is a compact type 6 triod. By definition there exist a continuum N and three points X, Y, and Z not belonging to N such that T is the sum of three continua H, K, and L irreducible from each point of N to X, Y, and Z, respectively. It follows from the definition that N is a nucleus of T. Suppose that T has a nucleus N' distinct from N. Then there exists a triodic decomposition of T into continua H', K', and L' such that $H' \cdot K' = N'$. It follows from Theorem 1.6 that each of the continua H', K', and L' contains only one of the points X, Y, and Z. Suppose that H'

contains X, that K' contains Y, and that L' contains Z. It follows from Theorem 2.4 that N and N' intersect. Therefore $N \cdot H'$ exists. Denote by C the component of $H' - N \cdot H'$ which contains X. Then C is a subset of H, and hence $\bar{C}$ is a subcontinuum of H which contains X and a point of N. Therefore $\bar{C}$ is H and H' contains H. It can be shown in the same way that K' contains K and that L' contains L. Hence, under the supposition that N' and N are distinct, N' contains N as a proper subset. But this contradicts the fact that N' is a nucleus of T.

THEOREM 2.6. *If C is a connected subset of a compact triod T which intersects no nucleus of T then, for every triodic decomposition of T into continua H, K, and L, the set C intersects only one of the sets $H - H \cdot K$, $K - H \cdot K$, and $L - H \cdot K$.*

Proof. By Theorem 2.2 there exists a triodic decomposition of T into continua H', K', and L' which are subsets of H, K, and L, respectively, and such that $H' \cdot K' = N$ is a nucleus of T. Since $H' - N$, $K' - N$, and $L' - N$ are mutually separated and, by hypothesis, the connected set C does not intersect N, it follows that only one of the sets $C \cdot H'$, $C \cdot K'$, and $C \cdot L'$ exists. Hence, since $H - H \cdot K$, $K - H \cdot K$, and $L - H \cdot K$ are subsets of H', K', and L', respectively, C intersects only one of these sets.

THEOREM 2.7. *If N is a nucleus of a compact triod T and C is a component of $T - N$ which is an open subset of T and intersects no nucleus of T, then $\bar{C}$ is not a triod.*

Proof. Denote $T - (C + N)$ by E. Since C is an open subset of T, $E + N$ is a continuum. Suppose that there exists a triodic decomposition of $\bar{C}$ into continua H, K, and L. Denote $H \cdot K$ by M. At least one of the continua H, K, and L intersects N. Suppose that H does this. The set $K - M$ is not a subset of N, for if it were, K would be a subset of $H + L$. Let P be a point of $K - K \cdot (M + N)$. It follows similarly that there exists a point Q of $L - L \cdot (M + N)$. The sum of the continua $K + (H + N)$, $L + (H + N)$, and $E + (H + N)$ is T and the common part of each two of them is $H + N$. Furthermore $H + N$ is a proper subcontinuum of each of them since it contains neither P, Q, nor any point of E, and they therefore form a triodic decomposition of T. But, by Theorem 2.6, this is impossible since the complements of $H + N$ with respect to $K + (H + N)$ and $L + (H + N)$ contain points (P and Q) of C.

THEOREM 2.8. *If N is a nucleus of a compact, locally connected triod T and C is a component of $T - N$ which intersects no nucleus of T, then $\bar{C}$ is not a triod.*

Proof. Since T is a locally connected continuum, C is an open subset of T, and the theorem follows from the preceding one. .

THEOREM 2. 9. *If N is a nucleus of a compact unicoherent triod T and C is a component of $T - N$ which is an open subset of T and intersects no nucleus of T, then $\bar{C}$ is irreducible between N and some point.*

Proof. Suppose that $\bar{C}$ is not unicoherent. Then it is the sum of two continua H and K whose common part is not connected. Denote the continuum $T - C$ by E. The set $\bar{C} \cdot N$ is a continuum since it is the common part of the continua $\bar{C}$ and E whose sum is T. Suppose that $\bar{C} \cdot N + H \cdot K$ is not connected. If the continuum H intersects N but K does not, a contradiction is reached since T is the sum of the continua K and $E + H$ but $K \cdot (E + H) = H \cdot K$ is not connected. Similarly it is not true that K intersects N and H does not. Hence both H and K intersect N. But then, since T is the sum of the continua $E + H$ and $\bar{C} \cdot N + H \cdot K$, a contradiction is again reached since $(E + H) \cdot (\bar{C} \cdot N + H \cdot K) = \bar{C} \cdot N + H \cdot K$ which, by supposition, is not connected. Therefore $\bar{C} \cdot N + H \cdot K$ is connected. Since neither H nor K is a subset of N, it follows that there exist points P and Q belonging to $H - H \cdot (K + N)$ and $K - K \cdot (H + N)$, respectively. The sum of the three continua $E + H \cdot K$, $H + \bar{C} \cdot N$, and $K + \bar{C} \cdot N$ is T and the common part of each two of them is the continuum $\bar{C} \cdot N + H \cdot K$. Since it does not contain P, Q, nor any point of $E - N$, $\bar{C} \cdot N + H \cdot K$ is a proper subset of each of the continua $E + H \cdot K$, $H + \bar{C} \cdot N$, and $K + \bar{C} \cdot N$; these three continua thus form a triodic decomposition of T. But, by Theorem 2. 6, this is impossible since the complements of $\bar{C} \cdot N + H \cdot K$ with respect to the first two of these contain points (P and Q) of C. It follows that $\bar{C}$ is unicoherent.

Furthermore, by Theorem 2. 7, $\bar{C}$ is not a triod. It follows from Theorem 3. 2 [16] that $\bar{C}$ is irreducible between two of its points P and Q. Suppose that $\bar{C}$ is not irreducible from any point to a point of N. Let X be a point of $\bar{C} \cdot N$. There exist subcontinua H and K of $\bar{C}$ irreducible from X to P and Q, respectively. By supposition H does not contain Q, K does not contain P, and $\bar{C} = H + K$. Let E denote $T - C$. The sum of the continua $E + H$ and $\bar{C} \cdot N + K$ is T. Hence $\bar{C} \cdot N + H \cdot K$, their common part, is connected. With the use of this fact a contradiction may be established by an argument identical, word for word, with the one beginning with the words "The sum of the three continua" in the paragraph above. From this contradiction it follows that $\bar{C}$ is irreducible from N to some point.

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[16] This theorem is proved independently of Theorem 2. 9.

COROLLARY. *If there exist a nucleus N of a compact unicoherent triod T and three components of $T - N$ which are open subsets of T and intersect no nucleus of T, then T is a type 4 triod.*

COROLLARY. *If there exists a nucleus N of a compact unicoherent triod T such that $T - N$ has only a finite number of components, then if C is any component of $T - N$ which does not intersect a nucleus of T, $\bar{C}$ is irreducible from N to some point, and, if there are at least three such components, then T is a type 4 triod.*

THEOREM 2.10. *If N is a nucleus of a compact, unicoherent, locally connected triod T and C is a component of $T - N$ which intersects no nucleus of T, then $\bar{C}$ is an arc.*

That the condition that C be an open subset of T cannot be omitted in Theorems 2.7 and 2.9 is shown by the following example. Let M be a simple triod which is the sum of the arcs OX, OY, and OZ. Let $I_1, I_2, I_3, \cdots$ be an infinite sequence of arcs such that (1) X is an end point of each of them, (2) no one of them has any point except X in common with any other or with M, (3) each point of M is a limit point of $I_1 + I_2 + I_3 + \cdots$, and (4) $T = M + I_1 + I_2 + I_3 + \cdots$ is closed. The point X is the only nucleus of the triod T and $M - X$ is therefore a component of $T - X$ which intersects no nucleus of T. But $\overline{M - X} = M$ is a triod and hence not irreducible betwen any two points.

Theorems 2.9 and 2.10 do not follow without the condition of unicoherence. Consider a simple triod plus a simple closed curve which intersects it only in one of its end points. If, in the statement of Theorem 2.9, the condition of unicoherence were placed upon $\bar{C}$ instead of upon T it would follow that $\bar{C}$ is irreducible between two points but not that it is irreducible from N to some point.

According to the definition of a nucleus of a triod given above, it is possible that a given triod may have two nuclei, one a proper subset of the other. The definition may therefore be open to the objection that it is not restrictive enough and that it is too much dependent on the initial decomposition of the triod. The following definition eliminates the dependence on the initial decomposition.

Definition. The subcontinuum N of a triod T is said to be an absolute nucleus of T if there exists a triodic decomposition of T into continua H, K, and L such that $H \cdot K = N$ and such that there does exist a triodic decomposition of T into continua H', K', and L' such that $H' \cdot K'$ is a proper subset of N.

An absolute nucleus of a triod T is obviously a nucleus of T. A nucleus of a compact triod T is absolute if and only if it contains no other nucleus of T.

Example. There exists a triod which is a compact hereditary continuous curve in the plane but which has no absolute nucleus. Let space be the Cartesian plane. For each positive integer n and each non-negative integer k less than or equal to $(3^{n-1}-1)/2$ let R_{nk} be a square which has as a pair of opposite vertices the points of the x-axis having abscissas $2k/3^{n-1}$ and $(2k+1)/3^{n-1}$. Denote the vertex of R_{nk} having positive ordinate by P_{nk} and the vertex having negative ordinate by Q_{nk}. Denote by T the closure of the component of $R_{10}+R_{20}+R_{21}+\cdots$ which contains P_{10}; the continuum T is a triod. The common part of T and the x-axis is a Cantor middle third set C. Let U and V denote the sets of all points of T with non-negative and non-positive ordinates, respectively. Let N be any nucleus of T. Since T is not disconnected by any subcontinuum either of U or of V, it follows that N contains a point P of $U-C$ and a point Q of $V-C$. Since T is a hereditary continuous curve, N contains an arc PQ from P to Q. It follows from the fact that U and V are dendrons that PQ contains an arc AOB such that $AO-O$ is a subset of $U-C$, $BO-O$ is a subset of $V-C$, and if (x, y) are the co-ordinates of a point of AO, then the point with co-ordinates $(x, -y)$ belongs to BO. There exist integers n and k such that P_{nk} is a point of $AO-A$. Then Q_{nk} is a point of $BO-B$. But the interval $P_{nk}Q_{nk}$ of AOB is a nucleus of T and is a proper subset of N. It follows that T has no absolute nucleus.

THEOREM 2.11. *If N is a nucleus of a compact triod T such that $T-N$ is the sum of three connected sets, then N is an absolute nucleus of T.*

Proof. There exists a triodic decomposition of T into continua H, K, and L such that $H\cdot K=N$. The sets $H-N=U$, $K-N=V$, and $L-N=W$ are connected. Suppose that N is not an absolute nucleus of T. Then there exists a triodic decomposition of T into continua H', K', and L' such that $H'\cdot K'=N'$ is a proper subcontinuum of N. Denote by U', V', and W' the sets $H'-N'$, $K'-N'$, and $L'-N'$, respectively. Suppose that U' intersects $T-N$. Then it intersects one of the sets U, V, and W, say U. Since it is a connected subset of $T-N'$, U is a subset of U'. Hence if each of the sets U', V', and W' intersects $T-N$, it follows that each of them contains one of the sets U, V, and W. Suppose that U' contains U, that V' contains V, and that W' contains W. Then since it contains no point of $V+W$, H' is a subset of H. Similarly K' and L' are subsets of K and L, respectively. But this contradicts the fact that N is a nucleus of T. One of the sets U', V',

and W', therefore, contains no point of $T - N$. Suppose that U' does not. Then $U + V + W$ is a subset of $K' + L'$. Suppose that $N \cdot (K' + L')$ is not connected. Then it is the sum of two mutually exclusive closed sets E and F such that E contains N'. Hence N is the sum of the mutually separated sets F and $U' + E$, which is impossible since N is connected. It follows that $N \cdot (K' + L')$ is a continuum. Hence H, $V + N \cdot (K' + L')$, and $W + N \cdot (K' + L')$ are subcontinua of H, K, and L, respectively; their sum is T; and the common part of each two of them is $N \cdot (K' + L')$, a proper subcontinuum of N. But this contradicts the hypothesis that N is a nucleus of T, and the theorem follows.

That Theorem 2.11 does not remain true if the word "three" is replaced by the words "a finite number of" is shown by the following example. Let A and B be two points in the plane, and let AB be an arc from A to B. Let XAY and XBZ be mutually exclusive arcs whose intersections with AB are their interior points A and B, respectively. Then if $T = AB + XAY + WBZ$, AB is a nucleus of the triod T, but the only absolute nuclei of T are the points A and B. The following theorem will, however, be established.

THEOREM 2.12. *If N is a nucleus of a compact triod T such that $T - N$ is the sum of a finite number of connected sets, then N contains an absolute nucleus of T.*

Proof. There exists a triodic decomposition of T into continua H, K, and L such that $H \cdot K = N$. Suppose that N is not an absolute nucleus of T. Then there exists a nucleus M of T which is a proper subset of N. Suppose that some component of $T - M$ is a subset of N. Let G be the collection of all such components. Then $N - G^*$ is a continuum. The point set G^* contains no limit point of any component of $K - N$. Since, by hypothesis, the components of $K - N$ are finite in number, it follows that G^* contains no limit point of $K - N$, and hence that $(K - N) + (N - G^*) = K - G^*$ is a continuum. Similarly $L - G^*$ is a continuum. Therefore H, $K - G^*$, and $L - G^*$ are subcontinua of H, K, and L, respectively; their sum is T; and the common part of each two of them is $N - G^*$, a proper subcontinuum of N. But this involves a contradiction. It follows that every component of $T - M$ intersects, and therefore contains, some component of $T - N$. Hence $T - M$ does not have more components than $T - N$. Suppose that it has as many. A triodic decomposition of T may be formed in the following way. Let H', K', and L' denote the point sets obtained by adding together M and all the components of $T - M$ which contain components of $H - N$, $K - N$, and $L - N$, respectively. The continua H', K', and L' form a triodic decomposition of T such that $H' \cdot K' = M$, and they are subsets of H, K, and L,

respectively. But this involves a contradiction. Therefore $T - M$ has fewer components than $T - N$. If M is not an absolute nucleus of T, there exists a nucleus of T which is a proper subset of M, and the above process may be repeated. If n is the number of components of $T - N$, then after less than n repetitions there will be obtained a nucleus M' of T lying in N and such that $T - M'$ is the sum of three connected sets. It follows from Theorem 2. 11 that M' is an absolute nucleus of T, and the theorem is established.

Corollary. *Every compact type 5 triod has an absolute nucleus.*

That a type 4 triod need not necessarily have an absolute nucleus is shown by the following example. In a Cartesian plane for each positive integer n let I_n be the interval of the x-axis which has as end points the origin and the point with abscissa $1/n$. For each n let R_n be a point set topologically equivalent to a ray and such that (1) R does not intersect I_1 or R_k $(k \neq n)$, (2) each point of I_n is a limit point of R_n, and (3) $R_n + I_n$ and $T = I_1 + R_1 + R_2 + \cdots$ are closed. The type 4 triod T has no absolute nucleus.

Theorem 2. 13. *If F is any finite subset of a compact triod T and M is any subcontinuum of T which separates T into at least three components, then there exists a subtriod of T which contains F and has an absolute nucleus which is a subset of M.*

This theorem may be proven by means of an argument similar to the one used in the proof of Theorem 2. 1. That the theorem does not follow if the condition that F be finite is replaced by the condition that F be countable and closed is shown by the example which follows Theorem 2. 12.

3. Concerning continua which are not triods.

Theorem 3. 1. *If the locally compact, freely decomposable*[17] *continuum M fails to be a triod, then it is a continuous curve.*

Proof. Suppose that M is a freely decomposable continuum which is not a triod and which fails to be locally connected at the point P. Then $M - P$ either is connected or is the sum of two mutually separated connected sets. In either case there is a component C of $M - P$ such that $C + P$ is not locally connected at P. Since P is a non cut point of the freely decomposable con-

[17] The continuum M is said to be freely decomposable provided that if P and Q are distinct points of M, then M is the sum of two continua neither of which contains $P + Q$. This definition is given by F. B. Jones in his paper "Aposyndetic continua and certain boundary problems," *American Journal of Mathematics*, vol. 63 (1941), pp. 545-553, and is there shown by him to be, for locally peripherally compact continua, equivalent to semi-local connectedness.

tinuum $\bar{C} = C + P$, it may be shown that $\bar{C}$ is locally remotely connected at P. There therefore exists an open subset D of $\bar{C}$ containing P and such that (1) D contains no connected open subset of $\bar{C}$ containing P, and (2) $\bar{C} - D$ is a continuum. From the first of these two properties it follows that D is the sum of four mutually separated sets U, V, W, and L, L containing P. Since the sum of $\bar{C} - D$ and any one of these sets is connected, it follows that $M - (V + W)$, $M - (U + W)$, and $M - (U + V)$ are continua. Hence, by Theorem 1.2, M is a triod. But this involves a contradiction.

COROLLARY. *Every locally compact, atriodic, freely decomposable continuum is a simple continuous curve.*

The following theorem is a generalization of a theorem of R. L. Moore.[18]

THEOREM 3.2. *Every compact nondegenerate unicoherent continuum which is not a triod is irreducible between some two points.*

Proof. Suppose that there exists a compact nondegenerate unicoherent continuum M which is not a triod and which is not irreducible between any two points. It follows from Theorem 1.3 that M is not a type 1 triod. Since every indecomposable continuum is irreducible between some two points, M is the sum of two proper subcontinua H and K. Since M is unicoherent, $H \cdot K$ is a continuum. Denote $H - H \cdot K$ by U and $K - H \cdot K$ by V. Since M is not a triod, U and V are connected. It will now be shown that there exists a point A of U such that every subcontinuum of M which contains A and a point of K contains U. Suppose that no such point A exists.

Denote U by U_0 and the continuum $\bar{U} \cdot K$ by N_0. There exists a countable subset $C = P_1 + P_2 + P_3 + \cdots$ of U_0 such that $\bar{C} = \bar{U}_0$. Denote P_1 by A_1. There exists a subcontinuum H_1 of $\bar{U}_0$ which is irreducible from A_1 to N_0. Let $N_1 = N_0 + H_1$. If $\bar{U}_0 - N_1$ exists, denote it by U_1. It is connected. For if it were the sum of two mutually separated sets E and F, then M would be the sum of the three continua $K + N_1$, $N_1 + E$, and $N_1 + F$, which would violate the hypothesis that M is not a type 1 triod. Let n_2 denote the smallest integer n such that P_n belongs to U_1. Denote P_{n_2} by A_2. There exists a subcontinuum H_2 of $\bar{U}_1$ irreducible from A_2 to N_1. Let $N_2 = N_1 + H_2$ and let $U_2 = \bar{U}_1 - N_2$. If it exists, U_2 is connected. This process may be continued indefinitely unless, for some integer r, $U_0 - N_r$ does not exist. Suppose this to be the case and denote A_r by A. Let L be a subcontinuum of $\bar{U}_0$ which contains A and a point of N_0. Since H_r is irreducible from A to N_{r-1}, it is a

[18] R. L. Moore, "Concerning compact continua which contain no continuum that separates the plane," *Proceedings of the National Academy of Sciences*, vol. 20 (1934), pp. 41-45.

subset of L. Hence, under the supposition that L does not contain U_0, there exists a point of N_{r-1} not belonging to L. Then M is the sum of the continua K, L, and N_{r-1}, which have a point of N_0 in common. But since no one of them is a subset of the sum of the other two, a contradiction is reached since M is not a type 1 triod. It follows that the process described above is an infinite one. Two cases arise.

Case 1. Suppose that there exists an integer j such that for each integer i greater than j, N_j and $\bar{U}_i$ have a point in common. Suppose that the continuum $\bar{U}_j$ is the sum of two of its proper subcontinua E and F. Clearly neither E nor F contains U_j. The set $E.F$, therefore, does not intersect N_j, for if it did, M, which is the sum of the continua $K+N_j$, E, and F, would be a type 1 triod. It follows, then, that not both of the sets E and F intersect N_j, for if they did, M would be the sum of two continua, $K+N_j$ and $E+F$, whose common part is not connected. Let E be that one of the two sets which intersects N_j. Since F contains a point of C, it follows that for some integer k, N_k intersects F. Denote by Z the component of $N_k - N_k \cdot F$ which contains N_j. The continuum $\bar{Z}$ contains N_j and a point of F but no point of $F - E \cdot F$. The continuum M is the sum of the continua $K+\bar{Z}$, $F+\bar{Z}$, and $E+\bar{Z}$. These have a point in common, and clearly neither of the first two is a subset of the sum of the other two. The continuum $E+\bar{Z}$ is not a subset of the sum of the other two; for if every point of $E - E \cdot F$ were a point of $\bar{Z}$ and hence of N_k, then U_k would be a subset of F, and $\bar{U}_k$ and N_j would have no point in common. A contradiction results since M is not a type 1 triod. It follows that $\bar{U}_j$ is indecomposable. Since M is unicoherent and the sum of the continua $\bar{U}_j$ and $K+N_j$, $N_j \cdot \bar{U}_j$ is a continuum and hence is a subset of one of the composants of $\bar{U}_j$. Let A be a point of a different composant of $\bar{U}_j$. Suppose that there exists a proper subcontinuum L of $\bar{U}_0$ which contains A and a point of N_0. Since L contains A and a point of $N_j \cdot \bar{U}_j$, it contains $\bar{U}_j$. Hence, by supposition, there exists a point of $N_j - N_0$ not belonging to L. Then M is the sum of the continua K, L, and N_j which have a point in common and are such that no one of them is a subset of the sum of the other two. This involves a contradiction.

Case 2. For each integer j there exists an integer i such that N_j and $\bar{U}_i$ have no point in common. Some subsequence of the sequence $A_1, A_2, A_3, \cdots$ is convergent. Let A be the sequential limit point of one such sequence. Since for each n, A belongs to $\bar{U}_n$, it follows that it does not belong to $N_0 + N_1 + N_2 + \cdots$. Let L be a subcontinuum of $\bar{U}_0$ which contains A and a point of N_0. Suppose that for some integer j, N_j fails to be a subset of L. There exists an integer i such that $\bar{U}_i$ and N_j have no point in common.

The continuum M is the sum of the continua K, N_i, and $L + \bar{U}_i$. Since they have a point in N_0 in common, and no one of them is a subset of the sum of the other two, a contradiction is reached. It follows that L contains $N_0 + N_1 + N_2 + \cdots$ and hence the set C. But since $\bar{C} = \bar{U}_0$, it follows that $L = \bar{U}_0$.

Hence it has been shown that there exists a point A of U such that any subcontinuum of M which contains A and a point of K contains U. It follows in a similar way that there exists a point B of V such that any subcontinuum of M which contains B and a point of H contains V. Let W be a proper subcontinuum of M which contains A and B. It follows that W contains both U and V. Then there is a point of the continuum $H \cdot K$ which does not belong to W. Denote by Z the component of $W - \bar{U}$ which contains V. The continuum $\bar{Z}$ intersects $\bar{U}$. Hence M is the sum of three continua $\bar{Z}$, $H \cdot K$, and $\bar{U}$ which have a point of $\bar{U} \cdot K$ in common and are such that no one of them is a subset of the sum of the other two. But this leads to a contradiction since M is not a type 1 triod. This establishes the theorem.

It follows from Theorems 1. 4 and 3. 2 that in order for a compact nondegenerate unicoherent continuum to be irreducible between some two points, it is necessary and sufficient that it fail to be a triod. The following, which is a generalization of a theorem of N. E. Rutt,[19] is now easily established.

THEOREM 3. 3. *If for each point P of a compact unicoherent continuum M, M is the sum of two proper subcontinua each containing P, then M is a triod.*

Proof. The continuum M is not irreducible between any two of its points and hence, by virtue of Theorem 3. 2, is a triod.

THEOREM 3. 4. *Suppose that H and K are compact hereditarily decomposable continua whose common part is connected and each of which is irreducible between some two points. Then in order for $H + K$ to be irreducible between two points, it is necessary and sufficient that it fail to be a triod.*

Proof. The necessity of the condition follows from Theorem 1. 4. In the proof of the sufficiency, the case where one of the continua H and K is a subset of the other is trivial. Hence suppose that $H \cdot K = N$ is a proper subcontinuum of H and of K. Suppose that one of these continua, say H, is not irreducible from any point of N to any other point. Then there exist points A and B of $H - N$ such that H is irreducible from A to B. Let X

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[19] N. E. Rutt, "Some theorems on triodic continua," *American Journal of Mathematics*, vol. 56 (1934), pp. 122-132.

be a point of N. There exist subcontinua U and V of H irreducible from X to A and B, respectively. It follows that U does not contain B, V does not contain A, and that $U + V = H$. Since H is hereditarily decomposable, the set of all points P such that U is irreducible from A to P is a continuum U' not containing A. Similarly the set V' of all points P such that V is irreducible from B to P is a continuum not containing B. The set $U \cdot V$ is a subset of $U' + V'$. Hence the continua $U + (U' + V' + N)$, $V + (U' + V' + N)$, and $K + (U' + V' + N)$ form a triodic decomposition of $H + K$. It follows from this contradiction that there exist a point A, such that H is irreducible from A to a point X of N, and a point B such that K is irreducible from B to a point Y of N. Suppose that there exists a proper subcontinuum L of $H + K$ from A to B. It is easily seen that $U = L - L \cdot K$ is the same as $H - N$ and that $V = L - L \cdot H$ is the same as $K - N$. Hence U and V are connected. The continuum $\bar{U}$ is irreducible from A to each point of $\bar{U} \cdot N$, and it follows, therefore, with the aid of Theorem 116 of Chapter 1 of *Foundations*, that the set of all points P such that $\bar{U}$ is irreducible from A to P is a continuum H' such that $H' \cdot N = \bar{U} \cdot N$. Similarly there exists a subcontinuum K' of $\bar{V}$ which does not contain B and such that $K' \cdot N = \bar{V} \cdot N$. Since L is a continuum, it follows that there exists a continuum N' which lies in $N \cdot L$ and intersects both $\bar{U} \cdot N$ and $\bar{V} \cdot N$. Since L contains $H + K - N$, it follows from the supposition that L is a proper subcontinuum of $H + K$ that there is a point of N not belonging to $\bar{U} \cdot N + \bar{V} \cdot N + N'$. But then the continua $\bar{U} + (H' + K' + N')$, $\bar{V} + (H' + K' + N')$, and $N + (H' + K' + N')$ form a triodic decomposition of $H + K$. This involves a contradiction.

That the condition in the preceding theorem is not sufficient if the words "hereditarily decomposable" are omitted is shown by the following example. Let U, V, and W be three compact indecomposable continua which satisfy the following conditions: there exist three points O, P, and Q no two of which belong to the same composant of U, V, or W and such that $U \cdot V = O$, $U \cdot W = O + P$, and $V \cdot W = O + Q$. Then $U + W$ and $V + W$ are each irreducible between some two points, and their common part, W, is connected, but $U + V + W$ is neither irreducible between two points nor a triod. The following theorem, however, is easily established by a method of proof similar to that employed in the preceding theorem.

THEOREM 3.5. *Suppose that H and K are compact continua whose common part is connected and each of which is irreducible between some two points. Then in order for $H + K$ to be irreducible between two points, it is necessary and sufficient that it fail to be a type 1 triod.*

THEOREM 3.6. *Suppose that M is a compact unicoherent continuum and*

k is an integer greater than one such that M is separated by some subcontinuum into k components but is not separated by any subcontinuum into more than k components. Then M is irreducible about some k of its points and is not irreducible about any smaller number of points.

Proof. Let E be a subcontinuum of M such that $M - E$ is the sum of k components. Let C be any component of $M - E$. Suppose that $\bar{C}$ is the sum of two subcontinua H and K such that $H \cdot K$ is not connected. Both H and K intersect E, for if only K did, for example, M would be the sum of two continua, H and $K + (M - C)$, whose common part, $H \cdot K$, is not connected. The set $E + H \cdot K$ is connected, for if it were not, M would be the sum of two continua, $H + (M - C)$ and $K + (M - C)$, whose common part, $H \cdot K + (M - C)$, is not connected. But since both H and K contain points belonging neither to $H \cdot K$ nor to E, this leads to a contradiction as the continuum $E + H \cdot K$ separates M into more than k components. Hence $\bar{C}$ is unicoherent. Suppose that $\bar{C}$ is a triod. Then there exists a triodic decomposition of $\bar{C}$ into continua H, K, and L at least one of which, say L, intersects E. Since each of the sets H and K contains a point which belongs neither to L nor to E, it follows that $E + L$ separates M into more than k components. Hence $\bar{C}$ is not a triod. It follows from Theorem 3.2 that $\bar{C}$ is irreducible between some two of its points. If $\bar{C}$ is not indecomposable, it is the sum of two of its proper subcontinua H and K. It may easily be shown that only one of these, say K, intersects E. Since $\bar{C}$ is irreducible from a point A of $H - H \cdot K$ to a point of $K - H \cdot K$, it follows that the set $W = \overline{H - H \cdot K}$ is a continuum which is irreducible from A to each point of $F = W \cdot K$. If $\bar{C}$ is indecomposable it follows that the continuum $F = \bar{C} \cdot E$ is a subset of one of its composants. Hence if A is a point of a different composant of $\bar{C}$ and $\bar{C}$ is denoted by W, it follows that W is irreducible from A to each point of F. Application of the above results to each of the k components of $M - E$ shows that there exist k mutually exclusive subcontinua $W_1, W_2, \cdots, W_k$ of M such that for each n $(1 \leqq n \leqq k)$ there exist a point A_n and a continuum F_n such that $F_n = W_n \cdot \overline{(M - W_n)}$ and W_n is irreducible from A_n to each point of F_n. Suppose that there exists a proper subcontinuum L of M which contains $A_1 + A_2 + \cdots + A_k$. Since L contains a point of each of the sets $F_1, F_2, \cdots, F_k$, it contains all of the sets $W_1, W_2, \cdots, W_k$ and therefore fails to contain some point P of $M - (W_1 + W_2 + \cdots + W_k)$. It follows that $N = L - (W_1 + W_2 + \cdots + W_k) + (F_1 + F_2 + \cdots + F_k)$ is a subcontinuum of M. But since the points $P, A_1, A_2, \cdots, A_k$ lie in different components of $M - N$, this involves a contradiction.

UNIVERSITY OF CALIFORNIA AT LOS ANGELES.

THE SPACE OF METRICS ON A COMPACT METRIZABLE SPACE.*

By M. E. Shanks.

Let X be a compact metrizable space and consider all the metrics on X which preserve the topology. These metrics are particular continuous functions defined on the product space $X \times X = X^2$. Denote by $F(X)$ the Banach space [3, p. 53] of all continuous, symmetric, real functions on X^2 which vanish on the diagonal, normed in the usual manner, $\| f \| = \max | f(x, x') |$, where $x, x' \varepsilon X$, $f \varepsilon F(X)$. Let $M_0(X)$ be the subspace of $F(X)$ composed of all metrics on X, and $M(X)$ the closure of $M_0(X)$ in $F(X)$. It is the purpose of this paper to exhibit a number of properties of $M_0(X)$ and $M(X)$, some of which are analogous to those possessed by the space $C(X)$ of all continuous real functions on X.

Specifically, it was shown by Banach [3, p. 170] and later generalized by Stone [7] and Eilenberg [6], that X and Y are homeomorphic if and only if $C(X)$ and $C(Y)$ are congruent, where X and Y are compacta. The same theorem holds for $M(X)$. This result is easily deduced from the stronger statement that only the *isomorphism* of $M_0(X)$ and $M_0(Y)$ is required. This last theorem is the only one of this type known to the author.

The proofs of these theorems use methods different from those of Banach, Stone, or Eilenberg. The essential tool is the lattice $\mathfrak{D}(X)$ of all upper semi-continuous decompositions of X which has been considered by Arnold [9] and the author independently. Both $M(X)$ and $\mathfrak{D}(X)$ contain, in a sense, all continuous transformations of X.

Finally, it is shown that, if X is an arc, the closed linear extension of $M(X)$ is equal to $F(X)$. This yields some new universal metric spaces.

1. Metrics and continuous transformations. The space $M(X)$ contains, in addition to metrics on X, distance functions which are zero between distinct points of X but which satisfy the remaining postulates for a metric. These distance functions will be called *quasi-metrics.* Denote by θ the quasi-metric which is identically zero. It is easy to see that both $M_0(X)$ and $M(X)$ are what may be called semi-linear spaces. For, one has the ordinary addition

* Received August 5, 1943; presented in part, and with different proofs, to the American Mathematical Society, April 11, 1941.

of elements and multiplication by positive real numbers. The null element of $M(X)$ is θ. The space $M(X)$ is complete, while $M_0(X)$ is dense in $M(X)$.

A continuous transformation T which maps X onto a compact metric space Y generates an upper semi-continuous decomposition (u. s. c. d.) of X into disjoint closed sets $S_y = T^{-1}(y)$, where $y \varepsilon Y$. These u. s. c. d.'s have the characteristic properties (a) $\Sigma S_y = X$, (b) $\liminf S_{y_n} \cdot S_y \neq 0$ implies $\limsup S_{y_n} \subset S_y$, $(n = 1, \cdots)$. The sets S_y of the u. s. c. d. will be called *slices.* Slices consisting of but a single point will be called *trivial.* Conversely, every u. s. c. d. of X defines a continuous transformation of X onto a compact metrizable hyperspace Y whose "points" are the slices of the u. s. c. d. (For these results see e. g. [5, p. 98].)

Every quasi-metric ρ generates a decomposition D of X into disjoint closed sets on which $\rho(x, x') = 0$. That these sets are disjoint follows from the triangle inequality. Consider a sequence $\{S_n\}$, $(n = 1, 2, \cdots)$, of these sets and suppose that $\liminf S_n \cdot S_0 \neq 0$, where S_0 is an element of D. If $x_0 \varepsilon \liminf S_n \cdot S_0$ and $x \varepsilon \limsup S_n$, then $x \varepsilon \liminf S_{n_i}$, $(i = 1, 2, \cdots)$, where $\{S_{n_i}\}$ is some subsequence of $\{S_n\}$. Then also $x_0 \varepsilon \liminf S_{n_i}$ and since ρ is zero over an element of D, we have $\rho(x, x_0) \leqq \rho(x, S_{n_i}) + \rho(S_{n_i}, x_0)$, where $\rho(x, S_{n_i})$ is the least distance from x to S_{n_i}. But $\rho(x, S_{n_i})$ and $\rho(x_0, S_{n_i})$ both approach zero with $1/i$, so that $\rho(x, x_0) = 0$. Hence $x \varepsilon S_0$ and $\limsup S_n \subset S_0$. D is therefore an u. s. c. d. The hyperspace Y of the decomposition has $\rho(S_{y_1}, S_{y_2})$ for a metric.

Conversely, if T is a continuous transformation of X onto the compactum Y, with metric $\sigma(y, y')$, then $\rho(x, x') = \sigma(y, y')$, where $T(x) = y$ and $T(x') = y'$, is a quasi-metric on X. For suppose $\rho_0 \varepsilon M_0(X)$, then $t\rho_0 + \rho \varepsilon M_0(X)$ and approaches ρ as $t \to 0$. Thus, ρ is an accumulation element of $M_0(X)$, or $\rho \varepsilon M(X)$. This gives the following theorem.

THEOREM 1.1. *A decomposition D of the compact metrizable space X into disjoint closed sets is an u. s. c. d. if, and only if, it can be generated by a quasi-metric on X. Every quasi-metric on X generates a continuous transformation of X onto a compactum Y, whose points are the slices of D, and whose metric is $\sigma(y_1, y_2) = \rho(s_{y_1}, s_{y_2})$, and conversely.*

It now follows that if $T(X) = Y$, Y compact metrizable, then $M(Y)$ is congruently contained in $M(X)$ by the natural embedding described above, and this embedding preserves algebraic properties. However the equivalence of $M(Y)$ with a subset of $M(X)$ is not sufficient to make Y a continuous image of X as a simple example shows.

Most function spaces are lattices and the same is true of $M(X)$ if its elements are partially ordered in the natural manner by $>$, [2, p. 157]. With this ordering, *join* and *meet* are given respectively as follows: $\rho \vee \rho'(x,x') = \max[\rho(x,x'),\rho'(x,x')]$; $\rho \wedge \rho'(x,x') = \inf \sum_{i=1}^{n} \rho^*(x_{i-1},x_i)$, $x = x_0, x_1, \cdots, x_n = x'$, where $\rho^* = \min[\rho(x,x'),\rho'(x,x')]$. It is easily seen that $\rho \vee \rho'$ and $\rho \wedge \rho'$ satisfy the postulates for a metric, except for the non-vanishing between distinct points, and are therefore quasi-metrics by the argument preceding Theorem 1.1. On the other hand $\rho \vee \rho'$ is clearly a l. u. b. to ρ and ρ', and the g. l. b. must be less than or equal to ρ^*, but $\rho \wedge \rho'$ is the greatest function not greater than ρ^* which satisfies the triangle inequality. Thus $M(X)$ is a lattice. It is not, in general, modular.

2. The lattice of decompositions. While the semi-linear lattice $M(X)$ contains all u. s. c. d.'s of X, it has the defect of not having unique elements associated with a given u. s. c. d. The lattice of all u. s. c. d.'s of X does not have this fault and will now be introduced. It has been considered by Arnold [9] independently of the author and for more general spaces than are considered here.

Denote by $\mathfrak{D}(X) = \{D_\gamma\}$ the set of all u. s. c. d.'s of X. Partially order this set thus: $D_\gamma < D_\delta$ if each slice of D_δ is contained in some slice of D_γ. Then $\mathfrak{D}(X)$ contains a least element O, which has a single slice equal to X, and a greatest element I, all of whose slices are trivial.

Let $\{D_\alpha\}$, $\alpha \varepsilon A$, be any collection of elements of $\mathfrak{D}(X)$ and consider $S_x = \Pi_\alpha S_x{}^\alpha$, where $S_x{}^\alpha$ is the slice of D_α which contains the point x. This product is a closed set and two such sets are either identical or disjoint. The sum of all such distinct sets is X. Thus there is formed a decomposition D which we show is an u. s. c. d. Suppose S_x, S_{x_n}, $(n = 1, 2, \cdots)$, are slices of D and $\liminf S_{x_n} \cdot S_x \neq 0$. Then $\liminf S_{x_n}{}^\alpha \cdot S_x{}^\alpha \neq 0$ for all α, and $\limsup S_{x_n}{}^\alpha \subset S_x{}^\alpha$. This implies that $\limsup \Pi_\alpha S_{x_n}{}^\alpha \subset \Pi_\alpha S_x{}^\alpha = S_x$. Thus D is an u. s. c. d. Since S_x is the greatest set in all $S_x{}^\alpha$, D is a l. u. b., $D = V_\alpha D_\alpha$. Thus $\mathfrak{D}(X)$ is complete (in the lattice sense) with respect to joins.

Suppose that $D_\gamma, D_\delta \varepsilon \mathfrak{D}(X)$ and denote by $\{D_\alpha\}$ the set of all elements of $\mathfrak{D}(X)$ less than D_γ and D_δ. This set is not empty since it contains O. Consider $V_\alpha D_\alpha$. Since each slice $S_x{}^\alpha$ contains $S_x{}^\gamma$ and $S_x{}^\delta$, where the latter are the slices of D_γ, D_δ, containing x, their product $\Pi_\alpha S_x{}^\alpha$ contains $S_x{}^\gamma$ and $S_x{}^\delta$ and so $V_\alpha D_\alpha < D_\gamma, D_\delta$. Thus $D_\gamma \wedge D_\delta = V_\alpha D_\alpha$. $\mathfrak{D}(X)$ is therefore a lattice and since it has O and I and is complete with respect to joins it is also complete with respect to meets. (See [1, p. 17, Theorem 2.2].)

THEOREM 2.1. *$\mathfrak{D}(X)$ is a complete lattice.*

As will be seen shortly, $\mathfrak{D}(X)$ as a lattice determines X. The following theorem is of the same spirit and so is included here without proof, as it will not be used. For the terminology see [11, pp. 31-33].

THEOREM 2.2. *$\mathfrak{D}(X)$ is an inverse mapping system whose limit space is X.*

There are in $\mathfrak{D}(X)$ elements "covered" by I which are the lattice duals of "atoms," [1, p. 10]. These elements are called *dual-atoms* and will be represented by the pair of points which they identify, (x, x').

If an u. s. c. d. D has exactly one slice not reducing to a point, D will be said to be *simple.* Dual-atoms are simple.

THEOREM 2.3. *X is homeomorphic to Y if and only if $\mathfrak{D}(X)$ is lattice isomorphic to $\mathfrak{D}(Y)$.*

Half of the theorem is obvious. Suppose then that τ is a lattice isomorphism of $\mathfrak{D}(X)$ onto $\mathfrak{D}(Y)$, $\tau(\mathfrak{D}(X)) = \mathfrak{D}(Y)$. To get a one-one mapping of X onto Y consider the dual-atoms. Let $\{(x, x')\}$ be the set of all dual-atoms of $\mathfrak{D}(X)$, each identifying a given point x with some other point x'. These transform under τ into the set of all dual-atoms of $\mathfrak{D}(Y)$, each of which identifies a fixed point y of Y with some other point. To see this last statement, note that if two dual-atoms have a common point their meet is less than exactly three dual-atoms, while if they do not have a common point their meet is less than exactly two dual-atoms. There is thus defined a transformation T of X into Y, $T(x) = y$. Clearly T is "onto" Y and because of the isomorphism is one-one.

Before passing to the proof that T is continuous it may be remarked that it is possible to construct an "ideal" space homeomorphic to X from $\mathfrak{D}(X)$. The points of this space are certain maximal collections of dual-atoms with a common point. (See [9].)

For the proof of the continuity of T the following lemma will be needed.

LEMMA 2.1. *Suppose that $D \varepsilon \mathfrak{D}(X)$ is simple, then $\tau(D) = D'$ is also simple.*

If S is the non-trivial slice of D, let x be a fixed point of S and x' an arbitrary point of S. D then is the meet of all dual-atoms (x, x'). Thus D' is the meet of all $\tau((x, x')) = (y, y')$ and D' is simple.

Now, using the same notation as in the lemma, if S' is the non-trivial slice of D' then $T(S) = S'$ and T is a closed transformation. Thus T^{-1} is continuous and since Y is a compactum T is topological, which completes the proof.

Remark. The above proof uses only the isomorphism of the dual-atoms and simple u. s. c. d.'s. This will be needed in the proof of Theorem 3. 1.

To connect $\mathfrak{D}(X)$ and $M(X)$ define an equivalence relation "r" in $M(X)$ as follows: $\rho_1 \, r \, \rho_2$ if $\rho_1(x, x') = 0$ implies $\rho_2(x, x') = 0$ and conversely. The relation r splits $M(X)$ into equivalence classes $R_\gamma > R_\delta$ if there is a $\rho_\gamma > \rho_\delta$ where $\rho_\gamma \varepsilon R_\gamma$, $\rho_\delta \varepsilon R_\delta$.

THEOREM 2. 4. *The system $\{R_\gamma\}$ is isomorphic to $\mathfrak{D}(X)$.*

From Theorem 1. 1 it follows that each u. s. c. d. defines an equivalence class and conversely. This maps $\{R_\gamma\}$ one-one on $\mathfrak{D}(X)$. It is obvious that the ordering is the same.

3. The uniqueness of the spaces of metrics.

THEOREM 3. 1. *The compact metrizable spaces X and Y are homeomorphic if, and only if, there is a linear isomorphism between $M_0(X)$ and $M_0(Y)$.*

Clearly such an isomorphism exists if X and Y are homeomorphic. Suppose then that τ is a linear isomorphism of $M_0(X)$ onto $M_0(Y)$. Denote elements of $M_0(X)$ and $M_0(Y)$ by ρ and $\sigma = \tau(\rho)$ respectively. The following lemmas are needed.

LEMMA 3. 1. *τ may be extended to a linear isomorphism τ^* which maps $M(X)$ onto $M(Y)$.*

Suppose that $\rho \, \varepsilon \, M(X) - M_0(X)$ and that $\rho_0 \, \varepsilon \, M_0(X)$ is fixed, then $t\rho_0 + \rho \, \varepsilon \, M_0(X)$, $t > 0$, and $\tau(t\rho_0 + \rho) = \sigma_t \, \varepsilon \, M_0(Y)$. Now, for $s > t$, $\| \sigma_s - \sigma_t \| = \| \tau(s\rho_0 + \rho) - \tau(t\rho_0 + \rho) \| = \| \tau((s - t)\rho_0) \| = | s - t | \, \| \tau(\rho_0) \|$. Therefore $\lim \sigma_t = \sigma$, as $t \to 0$, exists. Define $\tau^*(\rho) = \sigma$. It is obvious that $\tau^* = \tau$ on $M_0(X)$.

To show that τ^* is a linear isomorphism, consider firstly $\tau^*(\rho + \rho') = \lim \tau(2t\rho_0 + \rho + \rho') = \lim [\tau(t\rho_0 + \rho) + \tau(t\rho_0 + \rho')] = \sigma + \sigma'$. Thus τ^* is additive. Secondly, since

$$\| \tau^*(\rho) \| = \| \lim \tau(t\rho_0 + \rho) \| = \lim \| \tau(t\rho_0 + \rho) \| \leqq \lim N \cdot \| t\rho_0 + \rho \| = N \| \rho \|,$$

where N is the norm of τ, [3, p. 54], τ^* has the same norm as τ and hence is continuous. Thirdly, τ^* is one-one, for if $\tau^*(\rho) = \tau^*(\rho')$ then $\tau^*(\rho_0 + \rho) = \tau(\rho_0 + \rho) = \tau^*(\rho_0 + \rho') = \tau(\rho_0 + \rho')$ which contradicts the isomorphism of τ. Finally, it is obvious that $\tau^*(M(X)) = M(Y)$ from the fact that τ is an isomorphism and the process works both ways.

LEMMA 3. 2. *Suppose that $\rho \varepsilon M(X)$ generates a simple u. s. c. d. which identifies exactly n points, $x_1, x_2, \cdots, x_n$, then so does $\sigma = \tau^*(\rho)$.*

The proof is by induction. Suppose that $n = 2$. Since $\sigma = \tau^*(\rho)$ is not a metric by Lemma 3. 1 it identifies at least two points of Y. Suppose that σ identifies three points y_1, y_2, y_3. There are three quasi-metrics $\sigma'_{12}, \sigma'_{23}, \sigma'_{31}$ identifying exactly y_1 and y_2, y_2 and y_3, y_3 and y_1 respectively. Then so also do $\sigma_{12} = \sigma'_{12} + \sigma$, $\sigma_{23} = \sigma'_{23} + \sigma$, $\sigma_{31} = \sigma'_{31} + \sigma$. The inverses of these latter under τ^* are such that $\rho_{12} - \rho$, $\rho_{23} - \rho$, $\rho_{31} - \rho$ are quasi-metrics and since $\rho_{12}, \rho_{23}, \rho_{31} > \rho$, each must identify x_1 and x_2. Their sum $\rho_{12} + \rho_{23} + \rho_{31}$ then does also. But $\sigma_{12} + \sigma_{23} + \sigma_{31} \varepsilon M_0(Y)$. This contradiction shows that σ cannot have more than two points in one slice.

A similar proof shows that σ cannot have more than one non-trivial slice and more generally that *if ρ generates a simple decomposition then so does σ.*

Suppose, then, that the assertion is true for $n = r$ and that ρ identifies exactly $r + 1$ points. Clearly σ cannot identify fewer than $r + 1$ points. Suppose that σ identifies $y_1, \cdots, y_{r+2}$. Choose σ'_k, $(k = 1, \cdots, (r + 2) \times (r + 1)/2)$, to identify each pair of $y_1, \cdots, y_{r+2}$ and let $\sigma_k = \sigma'_k + \sigma$; then σ_k identifies each pair of the y's. Then $\rho_k = \tau^{*-1}(\sigma_k)$ identifies a pair of $x_1, \cdots, x_{r+1}$ since $\rho_k = \rho'_k + \rho > \rho$. But at least two of these ρ_k, ρ_j, identify the same pair so that $\rho_k + \rho_j$ is a quasi-metric while $\sigma_k + \sigma_j$ is a metric.

This last proof serves also for the next lemma.

LEMMA 3. 3. *If ρ_1 and ρ_2 identify x_1 and x_2 and $\sigma_1 = \tau^*(\rho_1)$ identifies y_1 and y_2, then $\sigma_2 = \tau^*(\rho_2)$ does also.*

LEMMA 3. 4. *If ρ_1 identifies x_1 and x_2, and ρ_2 identifies x_2 and x_3, then σ_1 and σ_2 identify two pairs of points with a common point.*

Suppose that σ_1 and σ_2 identify y_1, y_2 and y_3, y_4, respectively, and choose a ρ_3 identifying x_1, x_2, x_3 exactly. If $\rho'_1 = \rho_3 + \rho_1$ and $\rho'_2 = \rho_3 + \rho_2$ then σ'_1 and σ'_2 identify y_1, y_2 and y_3, y_4, respectively, by Lemma 3. 3. Further, $\sigma_3 = \sigma'_1 - \sigma_1 = \sigma'_2 - \sigma_2$ identifies three points by Lemma 3. 2. But σ_3 identifies y_1, y_2 and y_3, y_4 which implies that some pair of y_1, y_2, y_3, y_4 are the same.

These lemmas show that under τ^* we have a correspondence between the

equivalence classes of dual-atoms and simple decompositions. Theorem 2.4, Theorem 2.3, and the remark to Theorem 2.3 now apply and the proof is complete.

It is now possible to get the Banach-Stone-Eilenberg type of theorem quite easily.

THEOREM 3.2. *The compact metrizable spaces X and Y are homeomorphic if and only if $M(X)$ is congruent to $M(Y)$. (Or $M_0(X)$ is congruent to $M_0(y)$.)*

If X and Y are homeomorphic, clearly such a congruence exists. Let then g be such a congruence, $g(M(X)) = M(Y)$. (If the congruence is between $M_0(X)$ and $M_0(Y)$ it is easy to show that this is an isomorphism, so Theorem 3.1 applies. Obviously the congruence of $M_0(X)$ and $M_0(Y)$ implies that of $M(X)$ and $M(Y)$.) By a theorem of Mazur and Ulam [3, p. 166] a congruence is linear if it carries the null-element into the null-element. It will be shown first, then, that $g(\theta) = \phi$, where θ and ϕ are the null-elements in $M(X)$ and $M(Y)$. In the proof of the theorem of Mazur and Ulam it is shown that a congruence carries the center of a pair into the center of a pair, that is $g(1/2(\rho_1 + \rho_2)) = 1/2(\sigma_1 + \sigma_2)$. (This result was proven for linear spaces whereas the quasi-metrics form only a semi-linear space, but the theorem remains valid since we may embed $M(X)$ in $F(X)$.) Thus if $g(\rho) = \phi$, $g(\theta) = \sigma$, and $g(\rho') = 2\sigma$ then $g(1/2(\rho + \rho')) = \sigma = g(\theta)$. Hence $1/2(\rho + \rho') = \theta$ which is impossible in a semi-linear space unless $\rho = \rho' = \theta$.

The linearity of g established it is only necessary, to complete the proof, to show that g carries $M_0(X)$ onto $M_0(Y)$ and apply Theorem 3.1. Suppose that $\rho_0 \varepsilon M_0(X)$ and that $\sigma_0 = g(\rho_0)$ identifies y_1 and y_2, where $\| \rho \| = \| \sigma_0 \| = 1$. Now choose σ of norm 1 so that $\sigma(y_1, y_2) = 1$. Then $\| t\sigma_0 - \sigma \| = 1$, for $0 < t < 1$. Hence $\| t\rho_0 - \rho \| = 1$. But $\rho(x, x') \leqq 1$ and $\rho_0(x, x') > 0$, $x \neq x'$, so that $| t\rho_0(x, x') - \rho(x, x') | < 1$ for all x, x', and since X is compact $\| t\rho_0 - \rho \| < 1$. This contradiction completes the proof.

4. Metrics on an arc. In this section X will always be the unit interval, $0 \leqq s \leqq 1$. Metrics on X are then special continuous symmetric functions on the square, $0 \leqq s \leqq 1$, $0 \leqq t \leqq 1$, which vanish on the diagonal. Because of symmetry it is necessary only to consider the upper half, Δ_0, of the square, $0 \leqq s \leqq t \leqq 1$. Let $M^*{}_0(x)$ be the linear extension of $M_0(X)$ in $F(X)$ and note that $M^*{}_0(X) \supseteq M(X)$. For if $\rho \varepsilon M(X)$ and $\rho_0 \varepsilon M_0(X)$ then $\rho_0 + \rho \varepsilon M_0(X)$ and $\rho = (\rho_0 + \rho) - \rho_0 \varepsilon M^*{}_0(X)$. Thus the linear extensions

of $M_0(X)$ and $M(X)$ are the same. Since $M(X)$ is complete it seems likely that $M^*_0(X)$ and $M^*(X)$ are complete but the author has been unable to show this.

THEOREM 4.1. *The closed linear extension* $\overline{M^*_0(X)}$ *is equal to* $F(X)$.

It will suffice to show that $M^*_0(X)$ is dense in $F(X)$. This is done by exhibiting a sequence of functions $\phi_n{}^k(s,t)$ in $M^*_0(X)$ whose linear combinations are dense in $F(X)$. We sketch the proof.

Define a sequence of subdivisions, $\{\Delta_n\}$, of the upper half, Δ_0, of the unit square. To get Δ_1, inscribe in Δ_0 a triangle whose vertices are the mid-points of the sides of Δ_0. This gives four triangles in Δ_1. If Δ_{n-1} is given, Δ_n is obtained by inscribing in each triangle of Δ_{n-1} a triangle whose vertices are the mid-points of its sides. The triangles of any subdivision, Δ_n, are all congruent isosceles right triangles. Consider the subdivisions up to Δ_n and denote by $p_n{}^k$, $(k=1,\cdots,m)$, the vertices of Δ_n *not* on the diagonal and *not* vertices of Δ_{n-1}.

Define the functions $\phi_n{}^k(s,t)$ on Δ_0 as follows: $\phi_n{}^k(s,t)=1$ at $p_n{}^k$; $\phi_n{}^k(s,t)=0$ on the complement of the combinatorial star of $p_n{}^k$; $\phi_n{}^k(s,t)$ is linear between $p_n{}^k$ and the boundary of its star.

It is easy to see that any function in $F(X)$ can be approximated as closely as one wishes by a linear combination of the $\phi_n{}^k$, [4, pp. 49, 51(d)]. To show that the $\phi_n{}^k$ are in $M^*_0(X)$ the following criterion for a metric will be used. Its proof is obvious.

LEMMA 4.1. *If* $f(s,t)\ \varepsilon\ F(X)$ *and* (a) $f(s,t')\geqq f(s,t)$ *for* $t'\geqq t\geqq s$, (b) $f(s',t)\leqq f(s,t)$ *for* $t\geqq s'\geqq s$, (c) $f(s,t), f(t,t')$ *are each greater than the planar metric equal to* $f(s,t')$ *at* (s,t'), *for all* $s\leqq t\leqq t'$, *then* $f(s,t)$ *is a metric.*

By a planar metric is meant merely a constant times the linear metric on X. Its graph in three space (s,t,z) is a plane. Now consider $\phi_n{}^k$. Let $\rho(s,t)$ be the planar metric equal to 1 at $p_n{}^k$ and define $\rho_1(s,t)$ equal to $\rho(s,t)$ if $\rho(s,t)\leqq 1$, and equal to 1 otherwise. $\rho_1(s,t)$ is then a metric. Let $\rho_2(s,t)=\rho_1(s,t)-2^{-n}\phi_n{}^k(s,t)$ on Δ_0. It is not hard to see that $\rho_2(s,t)$ satisfies (a), (b), and (c) of Lemma 4.1 and so ρ_2 is a metric. Since $\phi_n{}^k=2^n(\rho_1-\rho_2)\ \varepsilon\ M^*_0(X)$, the latter is dense in $F(X)$ and the proof is complete.

THEOREM 4.2. *The closed linear extension of* $M(Y)$ *is a universal metric space if the compact metrizable space* Y *has the power of the continuum.*

It was shown by Banach and Mazur [3, p. 187] that every separable metric space is congruently contained in the space C of all continuous real functions on an interval. Spaces with this property were called by Urysohn, [8], universal metric spaces.

If Y has the power of the continuum there is a continuous transformation of Y onto X and so $M(X)$ is congruently contained in $M(Y)$. Then $M^*(Y) \supset M^*(X)$ and it suffices to show that $\overline{M^*(X)} = F(X)$ is a universal metric space.

Consider all functions $f(0, t)$ such that $f(0, 0) = 0$, and extend these functions over Δ_0 by making the extension zero on the diagonal and linear between $(0, t)$ and (t, t). This set is equivalent to all continuous functions on $(0, 1)$ which vanish at 0 and, as an examination of the proof of Banach and Mazur shows, it is also a universal metric space.

THE UNIVERSITY OF MISSOURI.

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ON THE THEORY OF AUTOMORPHIC FUNCTIONS OF A MATRIX VARIABLE I—GEOMETRICAL BASIS.*

(*Dedicated to Professor K. L. Hiong, Chancellor of National Yunnan University, on his fiftieth birthday*)

By Loo-Keng Hua.

The present paper is a revised form of another manuscript which the author had previously submitted for publication. The revision was necessary because the original manuscript contained some results (found independently by the author in some research begun in 1941) that have been recently published in Prof. C. L. Siegel's paper on Symplectic Geometry.[1] It is the aim of this paper to give a brief account of those results which are interfluent with Siegel's contributions. The remaining part of the author's research will be given later separately.

The paper is divided into two parts: the first part (**1-7**) is algebraic in nature and gives a very brief description of the main theory with which the author deals. In the second part (**8-10**) the author proves that the spaces which play the important rôles in the theory of analytic mappings have non-positive Riemannian curvature. Thus the geometries under consideration are sufficiently regular, and the development (in broad-line) of the theory of automorphic functions presents no serious difficulties.

The situation of the problem is well described by a statement due to Poincaré:

Note.—Because of the poor mail service between the U. S. and China, a number of minor changes in this paper have been made here, with the consent of the editors, by Prof. Hua's friend Dr. Hsio-Fu Tuan and Prof. C. L. Siegel.

* Received September 10, 1943.

[1] C. L. Siegel, "Symplectic Geometry," *American Journal of Mathematics*, vol. 65 (1943), pp. 1-86. Another important reference is:

C. L. Siegel, "Einführung in die Theorie der Modulfunktionen n-ten Grades, *Math. Annalen*, vol. 116 (1939), pp. 617-657.

The author is greatly indebted to Prof. H. Weyl for sending him a copy of Siegel's paper on Symplectic Geometry. The author would like also to express his thanks to Prof. P. C. Tang and Prof. S. S. Chern, for each sent to him one of the following two important references:

G. Giraud, *Leçons sur les fonctions automorphes*, Gauthier-Villars, Paris, 1920;

E. Cartan, "Sur les domaines bornés homogènes de l'espace de n variables complexes," *Hamb. Abh.*, vol. 11 (1935), pp. 116-162.

"La géomètrie non-euclidienne est la clef véritable du probleme qui nous occupe," *Acta. Math.*, vol. 39 (1923), p. 100.

1. Groups. Throughout the paper, capital Latin letters denote $n \times n$ matrices with complex elements unless the contrary is stated. A' denotes the transposed matrix of A and $\bar{A}$ denotes the conjugate complex matrix of A. I denotes the unit matrix and O denotes the zero matrix.

We use the notations

$$\mathfrak{F} = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}, \qquad \mathfrak{F}_1 = \begin{pmatrix} O & I \\ I & O \end{pmatrix}$$

and

$$\mathfrak{I} = \begin{pmatrix} I & O \\ O & I \end{pmatrix}.$$

Let

$$\mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We shall consider three types of matrices $\mathfrak{T}$:

(i) Those $\mathfrak{T}$ satisfying

$$\mathfrak{T}\mathfrak{F}\mathfrak{T}' = \mathfrak{F}$$

are called *symplectic*. The condition may be written as

$$AB' = BA', \qquad CD' = DC', \qquad AD' - BC' = I;$$

(ii) Those $\mathfrak{T}$ satisfying

$$\mathfrak{T}\mathfrak{F}_1\mathfrak{T}' = \mathfrak{F}_1$$

are called *orthogonal*, or more definitely, $\mathfrak{F}_1$-*orthogonal*. The condition may be written as

$$AB' = -BA', \qquad CD' = -DC', \qquad AD' + BC' = I;$$

(iii) Those $\mathfrak{T}$ satisfying

$$\bar{\mathfrak{T}}\mathfrak{F}\mathfrak{T}' = \mathfrak{F}$$

are called *conjunctive-symplectic*. The condition may be written as

$$\bar{A}B' = \bar{B}A', \qquad \bar{C}D' = \bar{D}C', \qquad \bar{A}D' - \bar{B}C' = I.$$

Remark. Apparently we have a fourth type of matrices $\mathfrak{T}$ satisfying

$$\bar{\mathfrak{T}}\mathfrak{F}_1\mathfrak{T}' = \mathfrak{F}_1.$$

Since $\mathfrak{F}$ and $\mathfrak{F}_1$ are conjunctive the fourth type coincides with (iii).

THEOREM 1. *Each type of matrices form a group with respect to multiplication.*

Definition. The group formed by symplectic matrices is called the *symplectic group.* Similarly we define the $\mathfrak{F}_1$-orthogonal group and the *conjunctive-symplectic group.*

2. Spaces analogous to the projective space. (A detailed study will be given elsewhere later.)

DEFINITION 1. *A pair of matrices (Z_1, Z_2) is said to be symmetric or skew-symmetric if we have*

$$(Z_1, Z_2)\mathfrak{F}(Z_1, Z_2)' = O$$

or

$$(Z_1, Z_2)\mathfrak{F}_1(Z_1, Z_2)' = O$$

respectively.

(Certainly we might define Hermitian pairs, but this would be of no interest in the study of automorphic functions.)

DEFINITION 2. *A pair of matrices (Z_1, Z_2) is said to be non-singular, if the rank of the $n \times 2n$ matrix (Z_1, Z_2) is equal to n.*

DEFINITION 3. *A symplectic transformation is defined by*

$$(W_1, W_2) = Q(Z_1, Z_2)\mathfrak{T}$$

where Q is non-singular and $\mathfrak{T}$ is symplectic. Similarly, we define $\mathfrak{F}_1$-orthogonal and conjunctive-symplectic transformations.

THEOREM 2. *A symplectic transformation carries a non-singular symmetric pair of matrices into a non-singular symmetric pair. An $\mathfrak{F}_1$-orthogonal transformation carries a non-singular skew-symmetric pair of matrices into a non-singular skew-symmetric pair.*

Proof (for the symplectic case).

$$\begin{aligned}(W_1, W_2)\mathfrak{F}(W_1, W_2)' &= Q(Z_1, Z_2)\mathfrak{T}\mathfrak{F}\mathfrak{T}'(Z_1, Z_2)'Q' \\ &= Q(Z_1, Z_2)\mathfrak{F}(Z_1, Z_2)'Q' = O.\end{aligned}$$

Thus we may take a non-singular symmetric pair (Z_1, Z_2)[2] as a point of the space and the symplectic group as the group of motions of the space. Then

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[2] Identify (Z_1, Z_2) with (QZ_1, QZ_2) for any non-singular Q.

we obtain a geometry analogous to the projective geometry. A similar consideration holds for skew-symmetric pairs. A detailed treatment of independent interest will be given elsewhere.

We can also take a non-singular pair (Z_1, Z_2) as a point of our space and the conjunctive-symplectic group as the group of motions of the space. Then we also obtain a type of geometry.

We now indicate a general treatment which will be described for the symplectic case only.

Let $\mathfrak{H}$ be a Hermitian matrix. We define the symmetric pairs making

$$(\overline{Z_1, Z_2})\mathfrak{H}(Z_1, Z_2)'$$

positive definite to be a space.[3] The group of motions of the space is the subgroup of the symplectic group leaving $\mathfrak{H}$ invariant. Thus we establish a geometry analogous to non-euclidean geometry.

Thus the symplectic classification of Hermitian matrices is of the first importance. After the classification and the study of the structure of the group of automorphisms we arrive at the conclusion that there are three types of geometries of fundamental importance, namely those with

$$(1) \qquad \mathfrak{H} = \begin{pmatrix} H & O \\ O & H \end{pmatrix},$$

where H is a diagonal matrix $[1, \cdots, 1, -1, \cdots, -1]$; those with

$$(2) \qquad \mathfrak{H} = \begin{pmatrix} H & O \\ O & O \end{pmatrix};$$

and those with

$$(3) \qquad \mathfrak{H} = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}.$$

Correspondingly, we define the related geometries to be elliptic with signature H, parabolic with signature H and hyperbolic, respectively.

In this paper, we consider only the hyperbolic geometry. More definitely, all non-singular pairs (Z_1, Z_2) of matrices, making

$$(\overline{Z_1, Z_2}) \begin{pmatrix} I & O \\ O & -I \end{pmatrix} (Z_1, Z_2)'$$

positive definite, form a hyperbolic space. The symplectic transformations with matrix $\mathfrak{T}$ satisfying

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[3] It is defined to be a hypercircle in the "projective" space.

$$\bar{\mathfrak{T}} \begin{pmatrix} I & O \\ O & -I \end{pmatrix} \mathfrak{T}' = \rho \begin{pmatrix} I & O \\ O & -I \end{pmatrix}, \qquad \rho = \pm 1$$

form the group of motions of the space.

Analogously, the non-singular skew-symmetric pairs (Z_1, Z_2) of matrices making

$$(\bar{Z}_1, \bar{Z}_2) \begin{pmatrix} I & O \\ O & -I \end{pmatrix} (Z_1, Z_2)'$$

positive definite form a space. The $\mathfrak{F}_1$-orthogonal transformations $\mathfrak{T}$ satisfying

$$\bar{\mathfrak{T}} \begin{pmatrix} I & O \\ O & -I \end{pmatrix} \mathfrak{T}' = \rho \begin{pmatrix} I & O \\ O & -I \end{pmatrix}$$

form the group of motions of the space.

There is a distinction between the symplectic and the conjunctive-symplectic cases, since the introduction of a "hypercircle" is not necessary in the latter case. The non-singular pairs of matrices (Z_1, Z_2) making

$$(\overline{Z_1, Z_2}) \begin{pmatrix} O & I \\ -I & O \end{pmatrix} (Z_1, Z_2)'$$

positive definite form the space, and the conjunctive-symplectic group is the group of motions of the space.

Notice that the transformation

$$(W_1, W_2) = Q(Z_1, Z_2) \begin{pmatrix} iI/\sqrt{2} & iI/\sqrt{2} \\ -iI/\sqrt{2} & iI/\sqrt{2} \end{pmatrix}$$

carries the space of points (Z_1, Z_2) such that

$$\text{“}(\overline{Z_1, Z_2}) \begin{pmatrix} I & O \\ O & -I \end{pmatrix} (Z_1, Z_2)' \text{ is positive definite”}$$

into the space of points (W_1, W_2) such that

$$\text{“}(\overline{W_1, W_2}) \begin{pmatrix} O & I \\ I & O \end{pmatrix} (W_1, W_2)' \text{ is positive definite.”}$$

3. An extension of the conjunctive-symplectic group. We now consider the conjunctive-symplectic case with

$$\mathfrak{H} = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}.$$

It is clear that there is no essential reason to restrict $\mathfrak{H}$ to be a Hermitian

matrix with signature (n, n), except to have an analogy with the symplectic case. Thus we may extend the conjunctive-symplectic case much further. The procedure is as follows:

Let

$$\mathfrak{H} = \begin{pmatrix} I^{(n)} & O \\ O & -I^{(m)} \end{pmatrix}.$$

The points of the space are then given by matrices

$$(Z_1^{(n)}, \qquad Z_2^{(n,m)})\ [4]$$

such that

$$(\overline{Z_1^{(n)}, Z_2^{(n,m)}})\mathfrak{H}(Z_1^{(n)}, Z_2^{(n,m)})'$$

is positive definite. The group of motions consists of the transformations

$$(W_1^{(n)}, W_2^{(n,m)}) = Q(Z_1^{(n)}, Z_2^{(n,m)})\mathfrak{T}^{(n+m)}$$

where

$$\bar{\mathfrak{T}}^{(n+m)} \begin{pmatrix} I^{(n)} & O \\ O & -I^{(m)} \end{pmatrix} \mathfrak{T}' = \begin{pmatrix} I^{(n)} & O \\ O & -I^{(m)} \end{pmatrix}.$$

Writing

$$\mathfrak{T} = \begin{pmatrix} A^{(n)} & B^{(n,m)} \\ C^{(m,n)} & D^{(m)} \end{pmatrix},$$

we have the conditions:

$$\bar{A}A' - \bar{B}B' = I, \qquad \bar{A}C' = \bar{B}D', \qquad \bar{C}C' - \bar{D}D' = I.$$

The group so obtained is called the conjunctive group of signature (n, m).

4. Non-homogeneous coördinates. Let (W_1, W_2) and (Z_1, Z_2) be two symmetric (or skew) pairs connected by

$$(W_1, W_2) = Q(Z_1, Z_2)\mathfrak{T}.$$

If W_1 and Z_1 are both non-singular, let

$$W = -W_1^{-1}W_2, \qquad Z = -Z_1^{-1}Z_2;$$

then W and Z are symmetric (or skew) matrices connected by

$$W = (-A + ZC)^{-1}(B - ZD),$$

[4] $Z^{(n)}$ denotes an $n \times n$ matrix: $Z^{(n,m)}$ denotes an $n \times m$ matrix.

i. e.,

$$Z = (AW + B)(CW + D)^{-1}.$$

Thus a non-singular symmetric pair of matrices (W_1, W_2) may be considered as the homogeneous coördinates of a symmetric (or skew) matrix W.

Now in non-homogeneous coördinates, the geometries take the following forms:

(i) The space is formed by the symmetric matrices Z satisfying

$$I - Z\bar{Z} > 0.$$[5]

The group of motions is given by

$$W = (AZ + B)(CZ + D)^{-1}$$

and

$$\mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is symplectic satisfying

$$\bar{\mathfrak{T}} \begin{pmatrix} I & O \\ O & -I \end{pmatrix} \mathfrak{T}' = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}.$$

(ii) The space is formed by the skew-symmetric matrices Z satisfying

$$I + Z\bar{Z} > 0.$$

The group of motions is given by

$$W = (AZ + B)(CZ + D)^{-1}$$

where $\mathfrak{T}$ is $\mathfrak{F}$-orthogonal leaving $I + Z\bar{Z} > 0$ invariant.

(iii) The space is formed by the $n \times m$ matrices Z satisfying

$$I - Z\bar{Z}' > 0.$$

The group of motions is the conjunctive group of signature (n, m).

As we transform $\begin{pmatrix} I & O \\ O & -I \end{pmatrix}$ into $\begin{pmatrix} O & I \\ I & O \end{pmatrix}$, we find that the symplectic and the conjunctive-symplectic cases have also the following equivalent expressions.

For the symplectic case, the space is formed by the symmetric matrices

$$Z = X + iY$$

[5] "> 0" means "being positive definite."

with Y positive definite, and the group of motions can be easily verified to be the real symplectic group.

For the conjunctive-symplectic case, we define

$$\frac{Z' + \bar{Z}}{2}, \qquad \frac{Z' - \bar{Z}}{2i}$$

to be the virtual real and imaginary parts of Z. The space is formed by the matrices Z with positive definite virtual imaginary parts. The group of motions is the conjunctive-symplectic group. Both correspond to the Poincaré half-plane.

Remark. After laying the foundation of the present theory, the author found in Cartan's paper that there are four general types (and two special types) of bounded symmetric spaces for analytic mappings. They are the previous types (i), (ii) and (iii), and a type studied by G. Giraud (in 1920) with the hyperabelian group. Thus the present treatment may be considered as complete in a certain sense.

5. Metrization of the space. Let $\mathfrak{H}$ be a Hermitian matrix

$$\mathfrak{H} = \begin{pmatrix} H_1 & \bar{L}' \\ L & H_2 \end{pmatrix}.$$

Let $\mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a symplectic matrix satisfying

$$\bar{\mathfrak{T}}\mathfrak{H}\mathfrak{T}' = \mathfrak{H}.$$

and set

$$H(Z) = \bar{Z}H_2Z + \bar{Z}L + \bar{L}'Z + H_1; \quad Z = (AW - B)(-CW + D)^{-1}.$$

Then

$$W = (D'Z + B')(C'Z + A')^{-1} = (ZC + A)^{-1}(ZD + B).$$

Therefore

$$\begin{aligned} H(W) = {} & \overline{(ZC + A)}^{-1}\{(\bar{Z}\bar{D} + \bar{B})H_2(D'Z + B') \\ & + (\bar{Z}\bar{D} + \bar{B})L(C'Z + A') + (\bar{Z}\bar{C} + \bar{A})\bar{L}'(D'Z + B') \\ & + (\bar{Z}\bar{C} + \bar{A})H_1(C'Z + A')\}(C'Z + A')^{-1} \\ = {} & \overline{(ZC + A)}^{-1}H(Z)(C'Z + A')^{-1}. \end{aligned}$$

Further

$$\begin{aligned} dW &= (ZC + A)^{-1}dZD - (ZC + A)^{-1}dZC\,(ZC + A)^{-1}(ZD + B) \\ &= (ZC + A)^{-1}\,dZ(ZC + A)'^{-1}. \end{aligned}$$

Therefore we have

THEOREM 3. *The characteristic equation of the matrix*

$$dZ(H(Z))^{-1}\, d\bar{Z}(\overline{H(Z)})^{-1}$$

is invariant under the group of automorphisms of the hypercircle $H(Z)$. In particular, we have

$$\sigma((H(Z))^{-1}\, d\bar{Z}(\overline{H(Z)})^{-1}\, dZ)$$

as an invariant quadratic differential form under the group, where $\sigma(X)$ denotes the trace of the matrix X.

For the case corresponding to the Poincaré half-plane, we have that the quadratic differential form

$$\sigma(Y^{-1}\, dZ Y^{-1}\, d\bar{Z})$$

is invariant under all real symplectic transformations.

A similar result holds for the $\mathfrak{F}$-orthogonal case.

THEOREM 4. *For the conjunctive group with signature (n, m), we have the invariant quadratic differential form*

$$\sigma((I - Z\bar{Z}')^{-1}\, dZ(I - \bar{Z}'Z)^{-1}\, d\bar{Z}'),$$

where

$$Z = Z^{(n,m)}.$$

Proof. Let

$$\mathfrak{T}^{(m+n)} = \begin{pmatrix} A^{(n,n)}, & B^{(n,m)} \\ C^{(m,n)}, & D^{(m,n)} \end{pmatrix}$$

be a matrix satisfying the condition

$$\bar{\mathfrak{T}} \begin{pmatrix} I^{(n)} & O \\ O & -I^{(m)} \end{pmatrix} \mathfrak{T}' = \begin{pmatrix} I^{(n)} & O \\ O & -I^{(m)} \end{pmatrix}.$$

This condition may also be written as

$$\begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} \begin{pmatrix} A' & -C' \\ -B' & D' \end{pmatrix} = I^{(m+n)}.$$

Consequently

$$\begin{pmatrix} A' & -C' \\ -B' & D' \end{pmatrix} \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} = I^{(m+n)},$$

i. e.,

$$A'\bar{A} - C'\bar{C} = I, \qquad A'\bar{B} = C'\bar{D}, \qquad -B'\bar{B} + D'\bar{D} = I.$$

On setting

$$W^{(n,m)} = (AZ^{(n,m)} + B)(CZ + D)^{-1},$$

we have

$$W = (Z\bar{B}' + \bar{A}')^{-1}(Z\bar{D}' + \bar{C}').$$

Then

$$\begin{aligned} I - W\bar{W}' &= I - (Z\bar{B}' + \bar{A}')^{-1}(Z\bar{D}' + \bar{C}')(D\bar{Z}' + C)(B\bar{Z}' + A)^{-1} \\ &= (Z\bar{B}' + \bar{A}')^{-1}((Z\bar{B}' + \bar{A}')(B\bar{Z}' + A) - (Z\bar{D}' + \bar{C}')(D\bar{Z}' + C))(B\bar{Z}' + A)^{-1} \\ &= (Z\bar{B}' + \bar{A}')^{-1}(I - Z\bar{Z}')(B\bar{Z}' + A)^{-1}. \end{aligned}$$

Furthermore

$$\begin{aligned} I - \bar{W}'W &= I - (\overline{CZ + D})'^{-1}(\overline{AZ + B})'(AZ + B)(CZ + D)^{-1} \\ &= (\overline{CZ + D})'^{-1}(I - \bar{Z}'Z)(CZ + D)^{-1}. \end{aligned}$$

Finally, we have

$$\begin{aligned} dW &= AdZ(CZ + D)^{-1} - (AZ + B)(CZ + D)^{-1}CdZ(CZ + D)^{-1} \\ &= (A - (Z\bar{B}' + \bar{A}')^{-1}(Z\bar{D}' + C')C)dZ(CZ + D)^{-1} \\ &= (Z\bar{B}' + \bar{A}')^{-1}dZ(CZ + D)^{-1}. \end{aligned}$$

Combining all these results, we have

$$\begin{aligned} &(I - W\bar{W}')^{-1}dW(I - \bar{W}'W)^{-1}d\bar{W}' \\ &\quad = (B\bar{Z}' + A)(I - Z\bar{Z}')^{-1}dZ(I - \bar{Z}'Z)^{-1}d\bar{Z}'(B\bar{Z}' + A)^{-1}. \end{aligned}$$

This completes the proof.

6. Unitary equivalence.

LEMMA. *Given a unitary matrix U, there exists a matrix V such that*

(a) $$V^2 = U, \quad V: \textit{unitary},$$

and

(b) $$\textit{if} \quad U'A = AU, \quad \textit{then} \quad V'A = AV.$$

Proof. There exists a unitary matrix Γ such that

$$\Gamma^{-1}U\Gamma = D,$$

where D is a diagonal matrix $[e^{i\theta_1}, \cdots, e^{i\theta_n}]$ and $0 \leq \theta_\nu < 2\pi$. Let

$$V = \Gamma D_{\frac{1}{2}}\Gamma^{-1},$$

where $D_{\frac{1}{2}} = [e^{\frac{1}{2}i\theta_1}, \cdots, e^{\frac{1}{2}i\theta_n}]$. Now we shall verify that V possesses the required properties. (a) is evident. For (b): If $U'A = AU$, then

$$\Gamma'^{-1}D\Gamma'A = A\Gamma D\Gamma^{-1},$$

i. e.,

$$D\Gamma'A\Gamma = \Gamma'A\Gamma D.$$

Thus D is commutative with $\Gamma'A\Gamma$; then so is $D_{\frac{1}{2}}$, i. e.,

$$D_{\frac{1}{2}}\Gamma'A\Gamma = \Gamma'A\Gamma D_{\frac{1}{2}}.$$

Consequently

$$V'A = AV.$$

THEOREM 5. *Let Z be a non-singular symmetric matrix with complex elements; then there exists a unitary matrix U such that*

$$UZU' = [\mu_1, \cdots, \mu_n]$$

where $\mu_1, \cdots \mu_n$ are the positive square roots of the characteristic roots of $Z\bar{Z}$.

Proof. Since $Z\bar{Z}$ is a positive definite Hermitian matrix, we have a unitary matrix V such that

$$VZ\bar{Z}\bar{V}' = [\mu_1^2, \cdots, \mu_n^2], \qquad \mu_\nu > 0;$$

i. e.,

(1) $$W\bar{W} = [\mu_1^2, \cdots, \mu_n^2],$$

where $W = VZV'$. Evidently, $W_0 = [\mu_1, \cdots, \mu_n]$ is a solution of (1).

Now

$$W\bar{W} = Wq_0\bar{W}_0, \quad \text{i. e.,} \quad (W_0^{-1}W)(\overline{W_0^{-1}W})' = I,$$

i. e., $W_0^{-1}W$ is unitary, say U_0; then

$$W = W_0U_0.$$

Since W and W_0 are both symmetric, we have

$$W_0U_0 = U'_0W_0.$$

By the lemma, we have a unitary matrix U such that $U^2 = U_0$ and $W_0U = U'W_0$. Then

$$W = W_0U_0 = W_0U^2 = U'W_0U.$$

Thus

$$VZV' = U'W_0U,$$

and we have proved the theorem.

THEOREM 6. *Let Z be a non-singular skew-symmetric matrix, then $Z\bar{Z}$ is a negative definite Hermitian matrix and its characteristic polynomial is a perfect square.*

Proof. Evidently, we have

$$Z\bar{Z} = -Z\bar{Z}',$$

hence $Z\bar{Z}$ is negative definite. Further

$$| Z\bar{Z} - \lambda I | = | \bar{Z} | \; | Z - \lambda\bar{Z}^{-1} |.$$

Since $Z - \lambda\bar{Z}^{-1}$ is a skew-symmetric matrix, its determinant is a perfect square.

THEOREM 7. *Let Z be a non-singular skew-symmetric matrix; then we have a unitary matrix U such that*

$$UZU' = \begin{pmatrix} O & d_1 \\ -d_1 & O \end{pmatrix} \dotplus \cdots \dotplus \begin{pmatrix} O & d_{n/2} \\ -d_{n/2} & O \end{pmatrix},$$

where $d_1^2, d_1^2, \cdots, d_{n/2}^2, d_{n/2}^2$ are the characteristic roots of $-Z\bar{Z}$.

Proof. By Theorem 6, we have a unitary matrix V such that

$$VZ\bar{Z}'\bar{V}' = [d_1^2, d_1^2, \cdots, d_{n/2}^2, d_{n/2}^2].$$

Let $VZV' = W$. Clearly

$$W_0 = \begin{pmatrix} O & d_1 \\ -d_1 & O \end{pmatrix} \dotplus \cdots \dotplus \begin{pmatrix} O & d_{n/2} \\ -d_{n/2} & O \end{pmatrix}$$

satisfies

$$W\bar{W}' = W_0\bar{W}'_0.$$

The remaining part of the proof is the same as that of Theorem 5.

Remark. Both Theorems 5 and 7 may be extended to the singular case without any essential difficulty.

7. Existence and uniqueness of the geodesic passing through two given points.

THEOREM 8. *Passing through any two points of the hyperbolic space with the symplectic group, there is one and only one geodesic.*

Before proceeding to prove this fundamental theorem, we shall require a theorem concerning the equivalence of point-pairs.

THEOREM 9. *In the hyperbolic space with the symplectic group (in Poincaré's representation) any two points are equivalent to the two points iI and iD, where D is a diagonal matrix*

$$[d_1, \cdots, d_n], \qquad d_\nu > 0.$$

Proof. 1) The group is evidently transitive. Thus we can let one of the two points be iI.

2) The transformations leaving iI fixed are all of the form

$$Z_1 = (AZ + B)(-BZ + A)^{-1},$$

i. e.,

$$\frac{Z_1 - iI}{Z_1 + iI} = (A + Bi)\left(\frac{Z - iI}{Z + iI}\right)(A' + B'i).$$

Since

$$I = AA' + BB' = (A + Bi)(A - Bi)',$$

we have $A + Bi$ unitary, and then the theorem follows from Theorem 5.

Proof of Theorem 8. 1) Without loss of generality, we may assume that the two points are

$$Ii \quad \text{and} \quad Di, \qquad D = [d_1, \cdots, d_n], \qquad d_\nu > 0.$$

Both points belong to the subspace $X = O$. Since

$$\begin{aligned} \sigma(Y^{-1}dZY^{-1}dZ) &= \sigma(Y^{-1}dXY^{-1}dX) + \sigma(Y^{-1}dY\,Y^{-1}dY) \\ &\geq \sigma(Y^{-1}dY\,Y^{-1}dY). \end{aligned}$$

and the equality holds only for $dX = O$, the geodesics connecting the two points all lie in the subspace $X = O$.

2) Further let

$$Y = M'\Lambda M$$

where

$$M = (\gamma_{ij}), \qquad (\gamma_{ij} = 0 \quad \text{for} \quad i > j, \quad \gamma_{ii} = 1),$$

and

$$\Lambda = [q_1, \cdots, q_n]$$

are obtained according to Jacobi's reduction of positive definite quadratic forms.

(Notice that the space with positive definite Y is mapped topologically into the space with $q_\nu > O$ in the new variables (q, γ)).

Then we have

$$\dot{Y} = \dot{M}'\Lambda M + M'\dot{\Lambda}M + M'\Lambda\dot{M},$$

$$Y^{-1}\dot{Y} = M^{-1}\Lambda^{-1}\dot{\Lambda}M + M^{-1}\Lambda^{-1}M'^{-1}\dot{M}'\Lambda M + M^{-1}\dot{M},$$

and

$$\sigma((Y^{-1}\dot{Y})^2) = \sigma((\Lambda^{-1}\dot{\Lambda})^2) + 2\sigma((M^{-1}\dot{M})^2) + 4\sigma(M^{-1}\Lambda^{-1}\dot{\Lambda}\dot{M}) + 2\sigma(M^{-1}\Lambda^{-1}M'^{-1}\dot{M}'\Lambda M),$$

since

$$\sigma(AB) = \sigma(BA) \quad \text{and} \quad \sigma(A') = \sigma(A).$$

Further, let

$$M^{-1} = (n_{hk}), \qquad n_{hk} = 0 \quad \text{for} \quad h > k,$$

then

$$\sigma((M^{-1}\dot{M})^2) = \sum_{i,j,k,l} n_{ij}\dot{m}_{jk}n_{kl}\dot{m}_{li} = \sum_{i \leq j \leq k \leq l \leq i} n_{ij}\dot{m}_{jk}n_{kl}\dot{m}_{li} = \sum (n_{ii}\dot{m}_{ii})^2 = 0,$$

since $\dot{m}_{ii} = 0$; and

$$\sigma(M^{-1}\Lambda^{-1}\dot{\Lambda}\dot{M}) = \sum n_{ij}q_j^{-1}\dot{q}_j\dot{m}_{ji} = \sum_{i \geqq j} n_{ij}q_j^{-1}\dot{q}_j\dot{m}_{ji} = 0.$$

Thus

$$\sigma(Y^{-1}\dot{Y}Y^{-1}\dot{Y}) = \sigma((\Lambda^{-1}\dot{\Lambda})(\Lambda^{-1}\dot{\Lambda})) + 2\sigma((\Lambda^{\frac{1}{2}}\dot{M}M^{-1}\Lambda^{-\frac{1}{2}})(\Lambda^{\frac{1}{2}}\dot{M}M^{-1}\Lambda^{-\frac{1}{2}})'$$
$$\geq \sigma((\Lambda^{-1}\dot{\Lambda})(\Lambda^{-1}\dot{\Lambda})'),$$

and the equality holds for $\dot{M} = O$. Therefore the geodesics connecting the two points lie in the subspace with $\dot{M} = O$, i. e., the subspace of real and diagonal Z.

The subspace of real and diagonal Z is Euclidean, since

$$\sigma((\Lambda^{-1}d\Lambda)(\Lambda^{-1}d\Lambda)' = \sum_{i=1}^{n} (d \log q_i)^2,$$

and we have proved the theorem.

Evidently, we may deduce

THEOREM 10. *All geodesics are symplectic images of the curves*

$$Z = i[\lambda_1^s, \cdots, \lambda_n^s], \qquad \lambda_\nu > 0$$

and $\sum_{\nu=1}^{n} \log^2 \lambda_\nu = 1$.

THEOREM 11. *The equations of the geodesics of the space are given by*

$$d^2Z/ds^2 + i(dZ/ds)Y^{-1}dZ/ds = O.$$

The previous results give a sufficient indication of the algebraic treatment. We shall now give a general treatment which seems to be a "direct" attack.

8. A general type of Riemannian geometry.

Let

$$S = S^{(n,m)} = (s_{ij}), \qquad T = T^{(m,n)} = (t_{ij}).$$

We consider the geometry with the Riemannian metric

$$\sigma((I^{(n)} - ST)^{-1}dS(I^{(m)} - TS)^{-1}dT).$$

Let

$$\Lambda_{ij} = (\partial/\partial s_{ij})S.$$

LEMMA. *If*

$$\sigma(\Lambda_{ij}S') = 0$$

for all i and j, then $S = O^{(n,m)}$.

The lemma is evident.

THEOREM 12. *The equations of the geodesics of the space are given by*

$$d^2I/ds^2 + 2(dT/ds)(I - ST)^{-1}(S - STS)(I - TS)^{-1}(dT/ds) = O,$$

$$d^2S/ds^2 + 2(dS/ds)(I - TS)^{-1}(T - TST)(I - ST)^{-1}(dS/ds) = O,$$

where I denotes, under evident circumstances, either $I^{(n)}$ or $I^{(m)}$.

Proof. We have

$$\begin{aligned}\partial\sigma/\partial s_{ij} &= \sigma\{(I - ST)^{-1}\Lambda_{ij}T(I - ST)^{-1}(dS/ds)(I - TS)^{-1}(dT/ds) \\ &\quad + (I - ST)^{-1}(dS/ds)(I - TS)^{-1}T\Lambda_{ij}(I - TS)^{-1}(dT/ds)\} \\ &= \sigma(\Lambda_{ij}[T(I - ST)^{-1}(dS/ds)(I - TS)^{-1}(dT/ds)(I - ST)^{-1} \\ &\quad + (I - TS)^{-1}(dT/ds)(I - ST)^{-1}(dS/ds)(I - TS)^{-1}T]).\end{aligned}$$

Next

$$(d/ds)(\partial\sigma/\partial s_{ij}) = (d/ds)(\sigma(\Lambda_{ij}(I-TS)^{-1}(dT/ds)(I-ST)^{-1}$$
$$= \sigma(\Lambda_{ij}[(I-TS)^{-1}(TdS/ds + (dT/ds)S)(I-TS)^{-1}(dT/ds)(I-ST)^{-1}$$
$$+ (I-TS)^{-1}(dT/ds)(I-ST)^{-1}(SdT/ds + (dS/ds)T)(I-ST)^{-1}$$
$$+ (I-TS)^{-1}(d^2T/ds^2)(I-ST)^{-1}]).$$

Thus

$$(d/ds)(\partial\sigma/\partial s_{ij}) - \partial\sigma/\partial s_{ij} = \sigma(\Lambda_{ij}(I-TS)^{-1}M(I-ST)^{-1},$$

where

$$M = d^2T/ds^2 + (dT/ds)(I-ST)^{-1}SdT/ds + (dT/ds)S(I-TS)^{-1}(dT/ds),$$

since

$$T(I-ST)^{-1} = (I-TS)^{-1}T, \text{ etc.}$$

The equations of the geodesics are

$$(d/ds)(\partial\sigma/\partial s_{ij}) - \partial\sigma/\partial s_{ij} = 0$$

for all i and j. We have, then,

$$d^2T/ds^2 + 2(dT/ds)(I-ST)^{-1}(S-STS)(I-TS)^{-1}(dT/ds) = 0,$$

since

$$(I-ST)^{-1}S + S(I-TS)^{-1}$$
$$= (I-ST)^{-1}(S(I-TS) + (I-ST)S)(I-TS)^{-1}$$
$$= 2(I-ST)^{-1}(S-STS)(I-TS)^{-1}.$$

Interchanging S and T, we have the other differential matrix-equation.

THEOREM 13. *The Riemannian curvature tensor of the space is given by*

$$\sigma(K(U,Y)K(U,Y) - K(U,Y)K(X,V)$$
$$- K(X,Y)K(U,V) + K(X,V)K(X,V))$$

where (U,V) and (X,Y) are two directions and

$$K(X,Y) = (I-ST)^{-1}X(I-TS)^{-1}Y.$$

Proof. (The method is borrowed from Siegel's paper). We write

$$R(S,T) = (I-TS)^{-1}T = T(I-ST)^{-1}$$
$$= (I-TS)^{-1}(T-TST)(I-ST)^{-1}.$$

Then

$$\begin{aligned}
dR(S,T) &= (I-TS)^{-1}dT + (I-TS)^{-1}(TdS + dTS)(I-TS)^{-1}T \\
&= (I-TS)^{-1}TdS(I-TS)^{-1}T + (I-TS)^{-1}dT(I + S(I-TS)^{-1}T) \\
&= (I-TS)^{-1}(TdST + dT)(I-ST)^{-1} \qquad (1)\\
&= R(S,T)dSR(S,T) + (I-TS)^{-1}dT(I-ST)^{-1}.
\end{aligned}$$

Now we define two covariant differentials $(\delta_1 U, \delta_1 V)$, $(\delta_2 U, \delta_2 V)$ by

$$\delta_i U = -UR(S,T)\delta_i S - \delta_i SR(S,T)U, \qquad i = 1, 2,$$

and $\delta_i V$ is defined similarly by interchanging S and T formally. Then

$$\begin{aligned}
\delta_1\delta_2 U &= -\delta_1 UR(S,T)\delta_2 S - U\delta_1 R(S,T)\delta_2 S \\
&\quad - \delta_2 S\delta_1 R(S,T)U - \delta_2 SR(S,T)\delta_1 U \\
&= (UR(S,T)\delta_1 S + \delta_1 SR(S,T)U)R(S,T)\delta_2 S \\
&\quad - U(R(S,T)\delta_1 SR(S,T) + (I-TS)^{-1}\delta_1 T(I-ST)^{-1})\delta_2 S \\
&\quad - \delta_2 S(R(S,T)\delta_1 SR(S,T) + (I-TS)^{-1}\delta_1 T(I-ST)^{-1})U \\
&\quad + \delta_2 SR(S,T)(UR(S,T)\delta_1 S + \delta_1 SR(S,T)U) \\
&= \delta_1 SR(S,T)UR(S,T)\delta_2 S + \delta_2 SR(S,T)UR(S,T)\delta_1 S \\
&\quad - U(I-TS)^{-1}\delta_1 T(I-ST)^{-1}\delta_2 S \\
&\quad - \delta_2 S(I-TS)^{-1}\delta_1 T(I-ST)^{-1}U.
\end{aligned}$$

We have, then,

$$\begin{aligned}
U^* = (\delta_1\delta_2 - \delta_2\delta_1)U &= U(P(\delta_1 S, \delta_2 T) - P(\delta_2 S, \delta_1 T)) \\
&\quad + (Q(\delta_1 S, \delta_2 T) - Q(\delta_2 S, \delta_1 T))U,
\end{aligned}$$

where

$$\begin{aligned}
P(A,B) &= (I-TS)^{-1}B(I-ST)^{-1}A, \\
Q(A,B) &= A(I-TS)^{-1}B(I-ST)^{-1}.
\end{aligned}$$

Similarly, interchanging S and T, we have

$$\begin{aligned}
V^* = (\delta_1\delta_2 - \delta_2\delta_1)V &= V(P^*(\delta_1 T, \delta_2 S) - P^*(\delta_2 T, \delta_1 S)) \\
&\quad + (Q^*(\delta_1 T, \delta_2 S) - Q^*(\delta_2 T, \delta_1 S))V,
\end{aligned}$$

where P^* and Q^* have obvious definitions.

We introduce a further covariant vector (X, Y). We have to evaluate

$$2R = \sigma(P(U^*, Y) + P(X, V^*)),$$

which is

$$\begin{aligned}
&\sigma\{(I-TS)^{-1}Y(I-ST)^{-1}U[P(\delta_1S,\delta_2T)-P(\delta_2S,\delta_1T)]\\
&\quad+U(I-TS)^{-1}Y(I-ST)^{-1}[Q(\delta_1S,\delta_2T)-Q(\delta_2S,\delta_1T)]\\
&\quad+(I-ST)^{-1}X(I-TS)^{-1}V[P^*(\delta_1T,\delta_2S)-P^*(\delta_2T,\delta_1S)]\\
&\quad+V(I-ST)^{-1}X(I-TS)^{-1}[Q^*(\delta_1T,\delta_2S)-Q^*(\delta_2T,\delta_1S)]\\
&=\sigma(P(U,Y)P(\delta_1S,\delta_2T)-P(U,Y)P(\delta_2S,\delta_1T)\\
&\quad+P(\delta_1S,Y)P(U,\delta_2T)-P(\delta_2S,Y)P(U,\delta_1T)\\
&\quad+P(\delta_2S,V)P(X,\delta_1T)-P(\delta_1S,V)P(X,\delta_2T)\\
&\quad+P(\delta_2S,\delta_1T)P(X,V)-P(\delta_1S,\delta_2T)P(X,V)).
\end{aligned}$$

Putting $\delta_1(S,T)=(U,V)$ and $\delta_2(S,T)=(X,Y)$, we obtain the Riemannian curvature tensor

$$\begin{aligned}
R=\sigma[&P(U,Y)P(U,Y)-P(U,Y)P(X,V)\\
&-P(X,Y)P(U,V)+P(X,V)P(X,V)].
\end{aligned}$$

Changing P into K, we obtain the result stated in the theorem.

9. A specialization. In particular, we put

$$S=Z,\qquad T=\bar{Z}',\qquad V=\bar{U}',\qquad X=\bar{Y}'$$

in the formula of Theorem 13. By the lemma below we then have

THEOREM 14. *The Riemannian curvature tensor of the space with the metric*

$$\sigma(K(dZ,d\bar{Z}'))$$

is equal to

$$\sigma((K(U,\bar{X}')-K(X,\bar{U}'))^2),$$

where

$$K(A,B)=(I-Z\bar{Z}')^{-1}A(I-\bar{Z}'Z)^{-1}B.$$

LEMMA. *Under the hypothesis of the theorem*

$$\sigma(K(U,\bar{U}')K(X,\bar{X}'))=\sigma(K(X,\bar{U}')K(U,\bar{X}')).$$

Proof. 1) The trace of the product of two Hermitian matrices A and B is real, since

$$\sigma(AB)=\sum_{r,s}a_{rs}b_{sr}=\sum\bar{a}_{sr}\bar{b}_{rs}=\sigma(\bar{A}\bar{B}).$$

2) We have

$$\sigma(K(U,\bar{U}')K(X,\bar{X}'))$$

$$=\sigma((I-\bar{Z}'Z)^{-1}\cdot\bar{X}'(I-Z\bar{Z}')^{-1}U(I-\bar{Z}'Z)^{-1}\bar{U}'(I-Z\bar{Z}')^{-1}X),$$

which is the trace of the product of two Hermitian matrices. Thus it is real.

3) Since

$$\sigma(\overline{(K(U,\bar{U}')K(X,\bar{X}'))'}) = \sigma(K(X,\bar{U}')K(U,\bar{X}')),$$

we have the theorem.

THEOREM 15. *The Riemannian curvatures of the three kinds of hyperbolic spaces are always non-positive.*

Proof. We have

$$K(U,\bar{X}') - K(X,\bar{U}')$$
$$= (I - Z\bar{Z}')^{-1}(U(I - \bar{Z}'Z)^{-1}\bar{X}' - X(I - \bar{Z}'Z)^{-1}\bar{U}') = (I - Z\bar{Z}')^{-1}M$$

where M is skew-Hermitian. Then the Riemannian curvature tensor is

$$R = -\sigma((I - Z\bar{Z}')^{-1}M(I - Z\bar{Z}')^{-1}\bar{M}').$$

For those points making $I - Z\bar{Z}'$ positive definite, we have a matrix P such that

$$(I - Z\bar{Z}')^{-1} = P\bar{P}';$$

then

$$R = -\sigma(T\bar{T}'),$$

where $T = \bar{P}'MP$. Thus R is non-positive at all points for all directions of the space.

We conclude consequently that passing through two points there is one and only one geodesic and that from a point we can draw a geodesic perpendicular to a given geodesic, etc.[6]

It may also be proved that the spaces are Einstein spaces, i. e., their Ricci tensors are proportional to the fundamental tensors. According to a result due to Schouten and Struik, the spaces cannot be conformal to Euclidean spaces.

10. A concluding remark. Theorem 14 and its consequences and the properties of the groups given in Cartan's paper lead to a neat generalization of the theory of automorphic functions.

NATIONAL TSING HUA UNIVERSITY.
INSTITUTE OF MATHEMATICS, ACAD. SINICA.

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[6] E. Cartan, *Leçons sur la géométrie des espaces de Riemann*, Paris, 1925, Note 3.